A new vanishing criterion for bounded cohomology

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Universidad Autónoma de Madrid

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The study of groups via algebraic invariants

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Invariants

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Define $H^n(\Gamma, E) = \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1})$.

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$\rightsquigarrow$ rigidity results, numerical constraints...

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Theorem (Burger-Monod '99)

Let Γ be an irreducible cocompact lattice in a higher rank Lie group. Then $c^n \colon H_b^2(\Gamma, E) \to H^2(\Gamma, E)$ is injective for any separable Hilbert space E with unitary Γ-action.

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Theorem (Bavard '91)

$H_b^2(\Gamma, \mathbb{R}) = 0 \implies scl(\Gamma) = 0$.

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Recent results and question

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Question

What algebraic property does this vanishing capture ?

Theorem (Campagnolo, Fournier-Facio, Lodha, Moraschini '23)

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- *If Γ is f. g. and has c. c., then Γ is abelian.*

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- $[H, t^m] = 1$ (if $m = \infty$, $t^m = 1$).

Definition (c. c.)

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Remark

- *If Γ is f. g. and has c. c., then Γ is abelian.*
- *c. c. and c. c. c. pass to quotients and derived subgroups.*

Theorem

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Theorem (Burger-Monod '99)

X an irreducible symmetric space of non-compact type of rank ≥ 3, $\Gamma \leq Isom(X)$ a cocompact lattice. Then

- *if X is not hermitian symmetric, then $H^2_b(\Gamma, \mathbb{R}) = 0$;*
- *if X is hermitian symmetric, then $H^2_b(\Gamma, \mathbb{R})$ is one-dimensional, generated by the Kähler class.*

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Applications

Stable mapping class group

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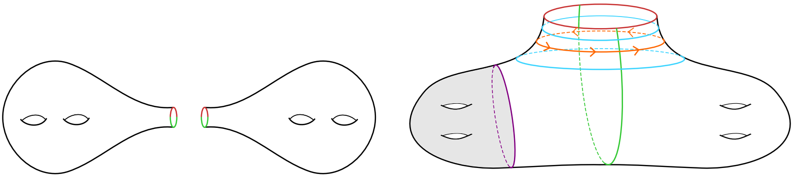

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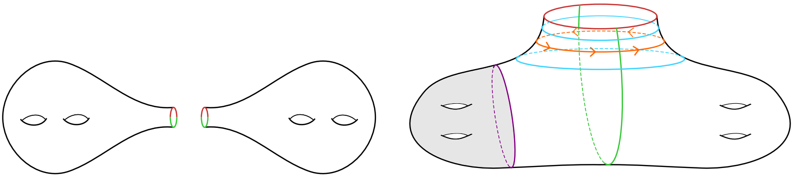

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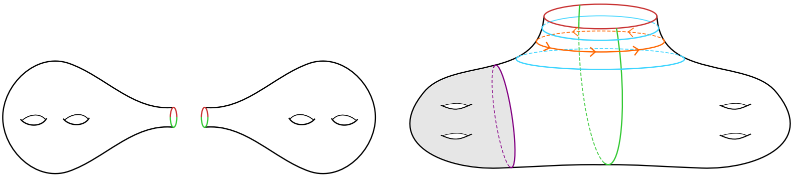

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Homeo-, diffeo- and symplectomorphisms groups with compact support

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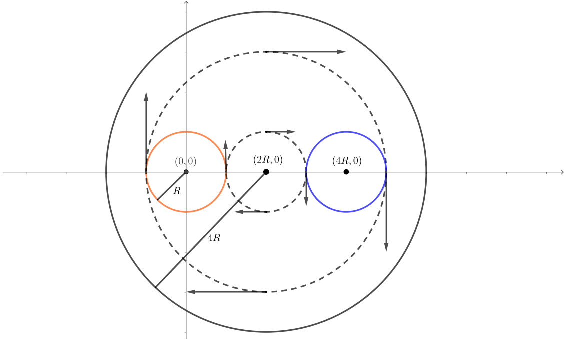

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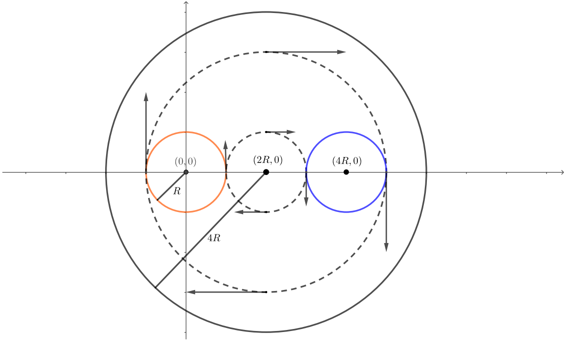

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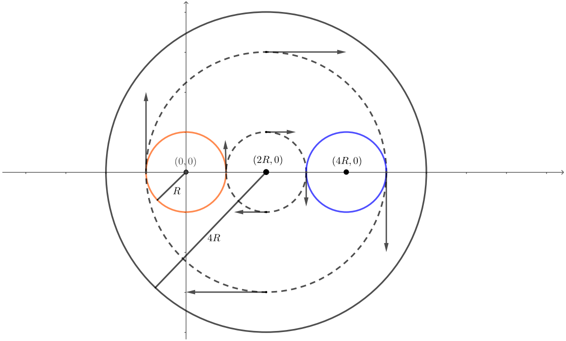

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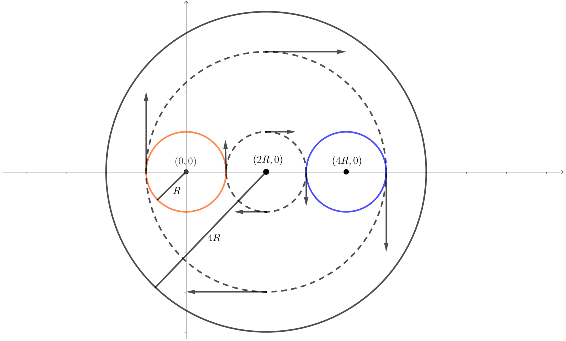

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Similar for $Sympl_c(\mathbb{R}^{2n} \times M)$, with M compact.

Ideas of proof (1)

c. c. c. and wreath products

Proposition (C.-F.-F.-L.-M. '23)

Γ has c. c. c.$\Longleftrightarrow \forall\, H \leq \Gamma$ f. g. subgroup, $\exists$ countable amenable group A, infinite transitive A-set X and homomorphism $f_H \colon H \wr_X A \to \Gamma$ s. t.

- *$f_H|_H \colon H \to \Gamma$ is the inclusion, and*
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$$\Gamma = \bigcup_{H \leq_{f.g.} \Gamma} H = \bigcup_{H \leq_{f.g.} \Gamma} Im(f_H).$$

Ideas of proof (2)

Wreath products and bounded cohomology

Proposition (Monod '22, C.-F.-F.-L.-M. '23)

If $\Gamma \cong H \wr_X A$ with

- *H, A countable ;*
- *A amenable ;*
- *X countable set, $A \curvearrowright X$, all orbits Ax are infinite ;*

then for any separable dual Banach Γ-module E, the complex of cochains

$$0 \longrightarrow E^\Gamma \xrightarrow{0} E^\Gamma \xrightarrow{id} E^\Gamma \xrightarrow{0} \dots$$

computes $H_b^*(\Gamma, E)$.

$$\Longrightarrow H_b^n(\Gamma, E) = 0 \quad \forall\, n > 0.$$

Ideas of proof (3)

Vanishing modulus

Definition

Γ group, E Banach Γ-module, $n > 0$. The *n-th vanishing modulus* of Γ with coefficients in E is

$$\|H_b^n(\Gamma, E)\| = \inf\{K \in [0, \infty] | \forall z \in Z_b^n(\Gamma, E), \exists b \in C_b^{n-1}(\Gamma, E)^\Gamma \text{ s. t. } d^{n-1}(b) = z, \|b\|_\infty \leq K\|z\|_\infty\}.$$

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If $\|H_b^n(\Gamma, E)\| < \infty$, $H_b^n(\Gamma, E) = 0$.

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Proposition (C.-F.-F.-L.-M. '23)

I index set, Γ group, $n \geq 1$.
Let $\{\Gamma_i\}_{i \in I}$ be a family of subgroups of Γ s. t. $\forall H \leq \Gamma$ f. g., $\exists i_0 \in I$ s. t. $H \leq \Gamma_{i_0}$.
$\implies \|H_b^n(\Gamma, E)\| \leq \sup_{i \in I} \|H_b^n(\Gamma_i, E)\|$ $\forall E$ *separable dual Banach Γ-module.*

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$\forall n \geq 1\, \exists K_n \in [0, \infty)$ s. t. $\forall \Gamma \cong H \wr_X A$ with

- *H, A countable;*
- *A amenable;*
- *X countable set, $A \curvearrowright X$, all orbits Ax are infinite;*

$\forall$ separable dual Banach Γ-module E, $\forall$ normal amenable $N \leq H \wr_X A$,

$$\|H_b^n((H \wr_X A)/N, E)\| \leq K_n.$$

Thank you for your attention !

Theorem (Monod '22)

- $\Gamma = F$ *Thompson's group* $\Longrightarrow H_b^*(\Gamma, E) = 0 \quad \forall * > 0, \forall$ *separable dual Banach* Γ*-module* E.
- $\Gamma = H \wr \mathbb{Z} = (\oplus_{n\in\mathbb{Z}} H) \rtimes \mathbb{Z} \Longrightarrow H_b^*(\Gamma, E) = 0 \quad \forall * > 0, \forall$ *separable dual Banach* Γ*-module* E.

Theorem (Monod-Nariman '23)

Let $j \geq 1, r \in \mathbb{N} \cup \{\infty\}$, *and* M *be a closed manifold. If* Γ *is*

- $\mathrm{Homeo}_c(M \times \mathbb{R}^j)$,
- $\mathrm{Diffeo}_c^r(M \times \mathbb{R}^j)$,
- $\mathrm{Homeo}_{c,0}(M \times \mathbb{R}^j)$,
- $\mathrm{Diffeo}_{c,0}^r(M \times \mathbb{R}^j)$,

$\Longrightarrow H_b^n(\Gamma, E) = 0 \quad \forall\, n > 0, \forall$ *separable dual Banach* Γ*-module* E.