Geometry and Topology Seminar
March 14, 2025, Lanzhou University

UNIVERSITÄT PADERBORN

Spectrum and Strichartz estimate on locally symmetric spaces

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Objects

(Continuous) Spectrum of Laplace-Beltrami operator

Temperedness of $L^2(\Gamma\backslash G)$

Critical exponent of discrete subgroup (Poincaré sereis)

Application (original motivation): Strichartz inequality

Hyperbolic Space

Hyperbolic plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\} \qquad \text{(upper half-plane)}$$

$$\mathbb{H}^2 = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$$

Real hyperbolic space

$$\mathbb{H}^n = \{x \in \mathbb{R} \times \mathbb{R}^n \,|\, -x_0^2 + x_1^2 + \cdots x_{n+1}^2 = -1,\ x_0 \geq 1\} \qquad \text{(hyperboloid)}$$

$$\mathbb{H}^n = \mathrm{SO}_e(n+1, \mathbb{R})/\mathrm{SO}(n)$$

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Noncompact symmetric space of rank 1

$$\mathbb{H}^n = \mathbb{H}^n(\mathbb{R}) \qquad \mathbb{H}^n(\mathbb{C}) \qquad \mathbb{H}^n(\mathbb{H}) \qquad \mathbb{H}^2(\mathbb{O})$$

L^2-Spectrum

- $-\Delta f = \lambda f$
- $\sigma(-\Delta) = \{\lambda \in \mathbb{C} \,|\, (-\Delta - \lambda)^{-1} : L^2 \to L^2 \text{ does not exist}\}$

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Euclidean setting

$\mathbb{R}^n$: $\sigma_c(-\Delta) = [0, \infty)$

$\to$ Torus $\mathbb{Z}^n \backslash \mathbb{R}^n$: $\sigma_d(-\Delta)$

$\to$ $\mathbb{Z}^{n-k} \backslash \mathbb{R}^n$: $\sigma_d(-\Delta) \cup \sigma_c(-\Delta)$

L^2-Spectrum

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Euclidean setting

$$\begin{tikzcd}[column sep=large] & \text{Torus } \mathbb{Z}^n \backslash \mathbb{R}^n\text{: } \sigma_d(-\Delta) \\ \mathbb{R}^n\text{: } \sigma_c(-\Delta) = [0, \infty) \arrow[ur] \arrow[dr] & \\ & \mathbb{Z}^{n-k} \backslash \mathbb{R}^n\text{: } \sigma_d(-\Delta) \cup \sigma_c(-\Delta) \end{tikzcd}$$

Real Hyperbolic space

Spectral gap

$\sigma_c(-\Delta)$

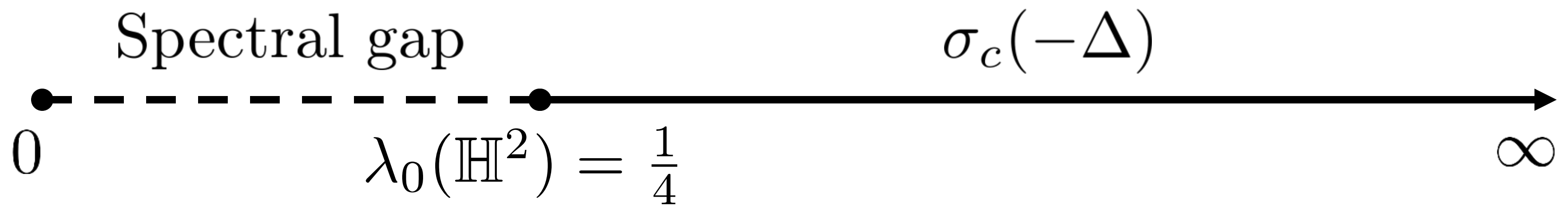
0

$\lambda_0(\mathbb{H}^n) = \left(\frac{n-1}{2}\right)^2$

∞

Hyperbolic Surface

Hyperbolic plane
$\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$:



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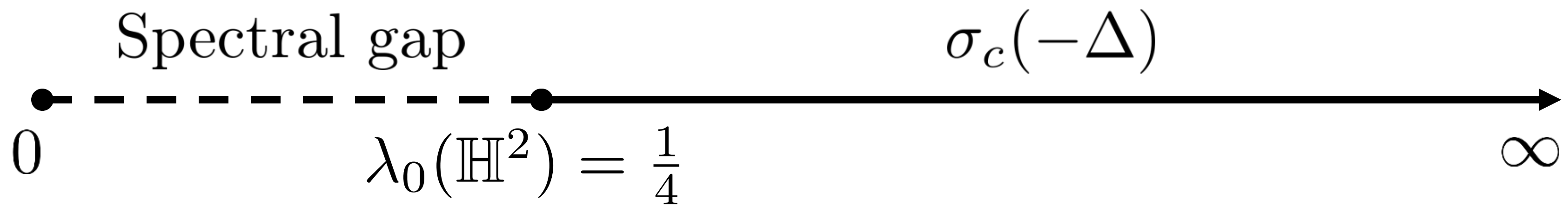

Modular curve $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$ (non-compact and finite area):

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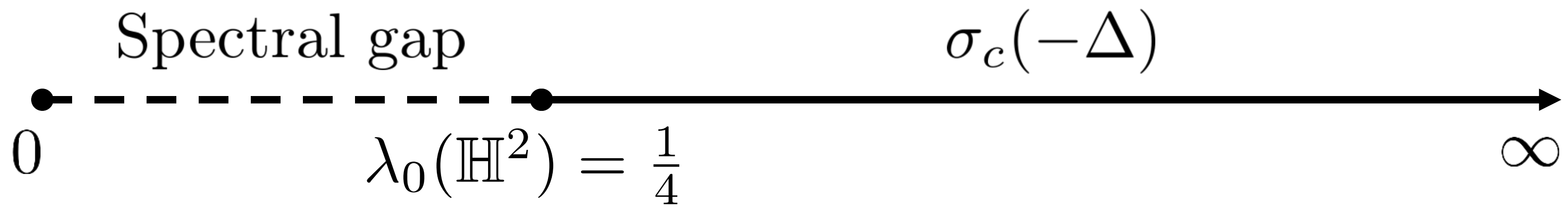

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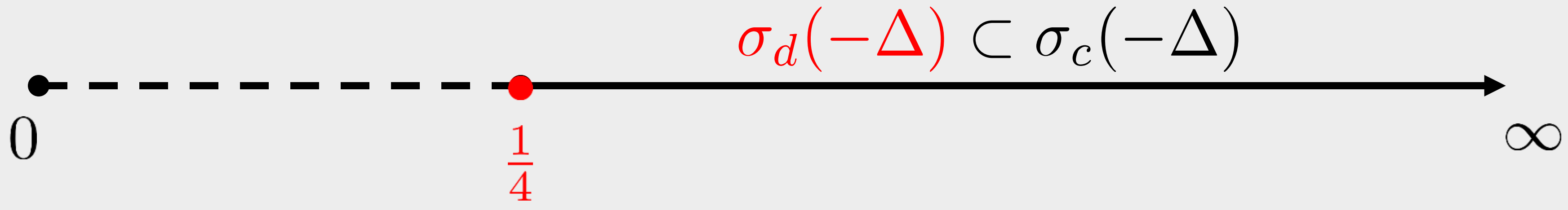

i.e., there are

- infinitely many embedded eigenvalues
- no exceptional eigenvalues

(Selberg's $1/4$ Conjecture for general Riemann surface)

Hyperbolic Surface

Hyperbolic plane
$\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$:

Spectral gap $\qquad \sigma_c(-\Delta)$

$0 \qquad \lambda_0(\mathbb{H}^2) = \frac{1}{4} \qquad \infty$

Thin group: $\Gamma \leq \mathrm{SL}(2,\mathbb{R})$ s.t. $\mathrm{Vol}(\Gamma\backslash\mathbb{H}^2) = \infty$

Hyperbolic Surface

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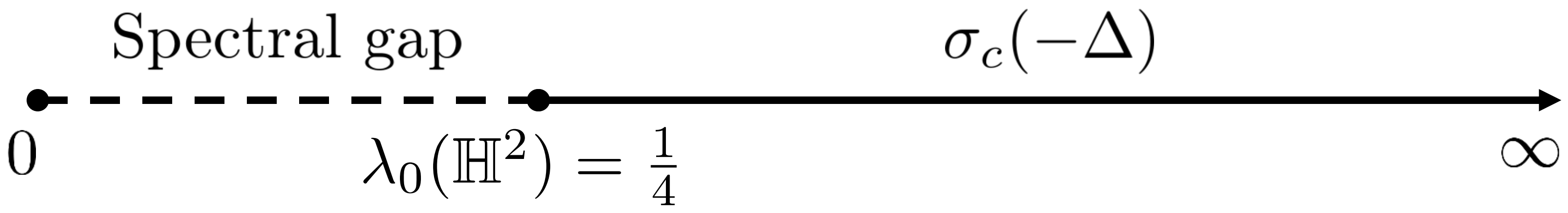

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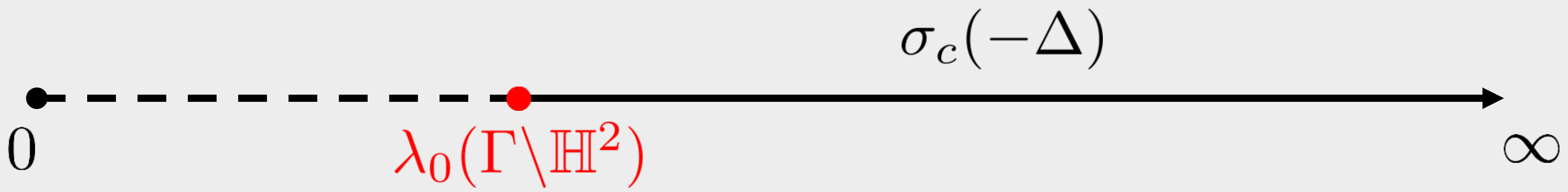

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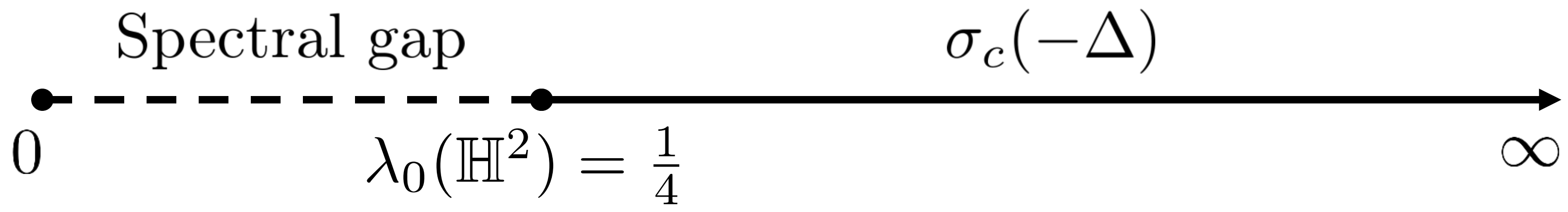

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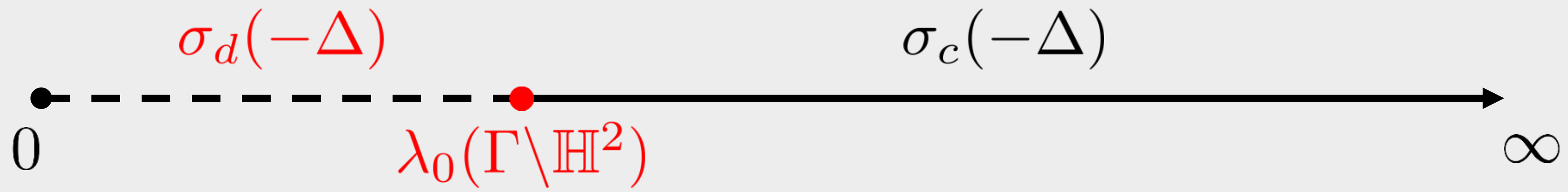

i.e., all eigenvalues are exceptional (finitely many)

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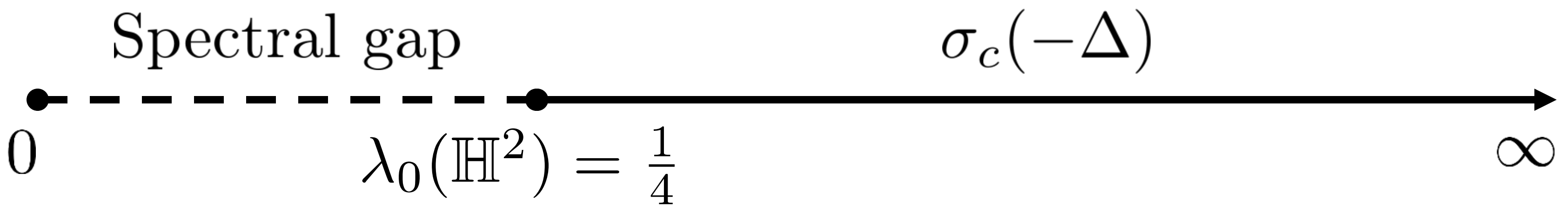

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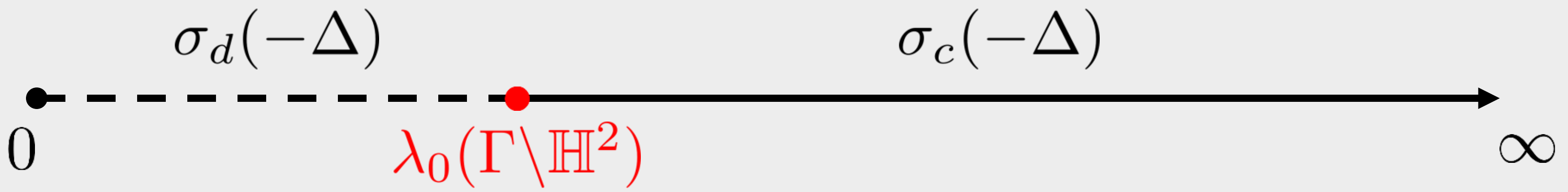

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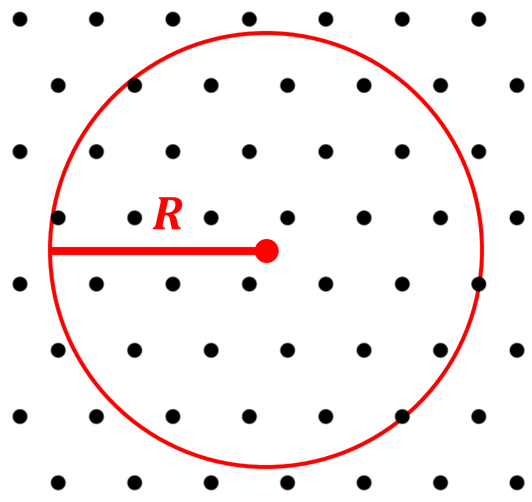
Characterize $$\lambda_0(\Gamma\backslash X) := \inf_{f \in C_c^\infty(\Gamma\backslash X)} \frac{\int_{\Gamma\backslash X} \|\mathrm{grad} f\|^2 \, d\mathit{vol}}{\int_{\Gamma\backslash X} \|f\|^2 \, d\mathit{vol}} = \inf \sigma_c(-\Delta)$$

Critical Exponent

$$\delta_\Gamma = \limsup_{R\to\infty} \frac{\log(\#\{\gamma \in \Gamma \mid d(e,\gamma e) \le R\})}{R}$$



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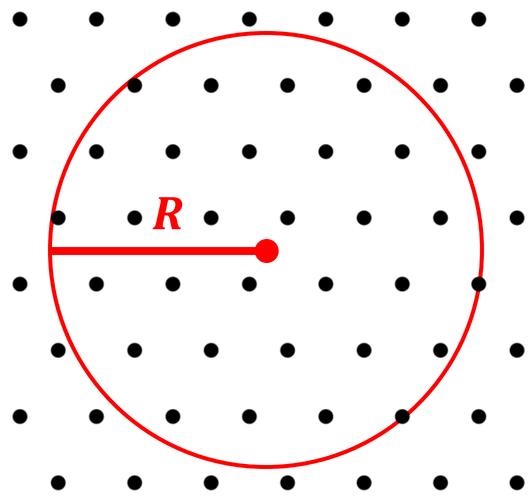

$$\delta_\Gamma = \inf \left\{ s \in \mathbb{R} \,\middle|\, \sum_{\gamma\in\Gamma} e^{-sd(e,\gamma e)} < \infty \right\}$$

Poincaré Series: $$\sum_{\gamma\in\Gamma} e^{-sd(e,\gamma e)} \begin{cases} < \infty & \text{if} \quad s > \delta_\Gamma \\ = \infty & \text{if} \quad s < \delta_\Gamma \end{cases}$$

e.g. In $\mathbb{H}^2$: $0 \leq \delta_\Gamma \leq 1$ In $\mathbb{H}^n$: $0 \leq \delta_\Gamma \leq n-1$

Characterization in Dimension 2

$\lambda_0(\Gamma\backslash\mathbb{H}^2) = \frac{1}{4}$

$L^2(\Gamma\backslash X)$ is tempered

$\delta(\Gamma) \leq \frac{1}{2}$

Theorem [Elstrodt '73 *Math. Ann.*, Patterson '76 *Acta Math.*]

$$\lambda_0(\Gamma\backslash\mathbb{H}^2) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq \delta_\Gamma \leq \frac{1}{2} \\ \frac{1}{4} - (\delta_\Gamma - \frac{1}{2})^2 & \text{if } \frac{1}{2} \leq \delta_\Gamma \leq 1 \end{cases}$$

Characterization on G/K of Rank 1

$$\lambda_0(\Gamma\backslash X) = \rho^2$$

$L^2(\Gamma\backslash X)$ is tempered

$$\delta(\Gamma) \leq \rho$$

Theorem [Elstrodt '73, Patterson '76, Sullivan '87 *JDG*, Corlette '90 *Invent. Math.*]

$$\lambda_0(\Gamma\backslash X) = \begin{cases} \rho^2 & \text{if } 0 \leq \delta_\Gamma \leq \rho \\ \rho^2 - (\delta_\Gamma - \rho)^2 & \text{if } \rho \leq \delta_\Gamma \leq 2\rho \end{cases}$$

where $\rho = \frac{n-1}{2}$ on $\mathbb{H}^n(\mathbb{R})$, n on $\mathbb{H}^n(\mathbb{C})$, $2n+1$ on $\mathbb{H}^n(\mathbb{H})$, 11 on $\mathbb{H}^2(\mathbb{O})$

Temperedness （柔曼性）

G connected semisimple Lie group $\implies$ direct integrals:

$$L^2(\Gamma\backslash G) \cong \int_{\widehat{G}}^{\oplus} \mathcal{H}_\pi \, \mathrm{d}\nu(\pi) \qquad \text{and} \qquad L^2(\Gamma\backslash X) \cong \int_{\widehat{G}_K}^{\oplus} (\mathcal{H}_\pi)^K \, \mathrm{d}\nu(\pi)$$

In rank 1, $\widehat{G}_K$ consists of

- the unitary spherical principal series $\pi_{\pm\lambda}$ $(\lambda \in \mathbb{R}/\pm 1)$
- the trivial representation $\pi_{\pm i\rho} = 1$
- the complementary series $\pi_{\pm i\lambda}$ $(\lambda \in I)$, where

$$I = \begin{cases} (0, \rho) & \text{if } X = \mathbb{H}^n(\mathbb{R}) \text{ or } \mathbb{H}^n(\mathbb{C}) \\ \left(0, \frac{m_\alpha}{2} + 1\right] & \text{if } X = \mathbb{H}^n(\mathbb{H}) \text{ or } \mathbb{H}^2(\mathbb{O}) \end{cases}$$

(no higher rank analogue)

Temperedness

$$
\begin{tikzcd}
\lambda_0(\Gamma\backslash X) = \rho^2 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & L^2(\Gamma\backslash G) \text{ is tempered} \\
& \delta(\Gamma) \leq \rho &
\end{tikzcd}
$$

By definition

- $L^2(\Gamma\backslash G)$ is called tempered if $\widehat{G}_K$ does not involve complementary series
- $-\Delta$ acts on $(\mathcal{H}_\pi)^K$ by multiplication by $\lambda^2 + \rho^2$

Question

$$\begin{tikzcd}
\lambda_0(\Gamma\backslash X) = \rho^2 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & L^2(\Gamma\backslash G) \text{ is tempered} \arrow[dl, leftrightarrow] \\
& \delta(\Gamma) \leq \rho &
\end{tikzcd}$$

? When $\Gamma\backslash X$ is of higher rank and infinite volume?

Noncompact (Riemannian) Symmetric Space

A noncompact symmetric space is a complete Riemannian manifold

- with nonpositive sectional curvature
- which is simply connected (Cartan-Hadamard manifold)

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- with nonpositive sectional curvature
- which is simply connected (Cartan-Hadamard manifold)
- with symmetric property
- which growths exponentially fast at infinity
- which can be identified as a homogeneous space G/K

e.g. $\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ and $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$

Locally Symmetric Space

Noncompact symmetric space

$$X = G/K$$

- G noncompact semisimple Lie group (connected, finite center)
- K maximal compact subgroup of G

- $\Gamma \leq G$: discrete and torsion-free subgroup of G
  - Γ is a lattice: $\mathrm{Vol}(\Gamma \backslash X) < \infty$
  - Γ has infinite covolume: $\mathrm{Vol}(\Gamma \backslash X) = \infty$

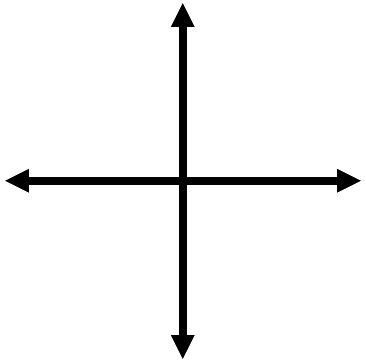
Weyl Chamber

Rank

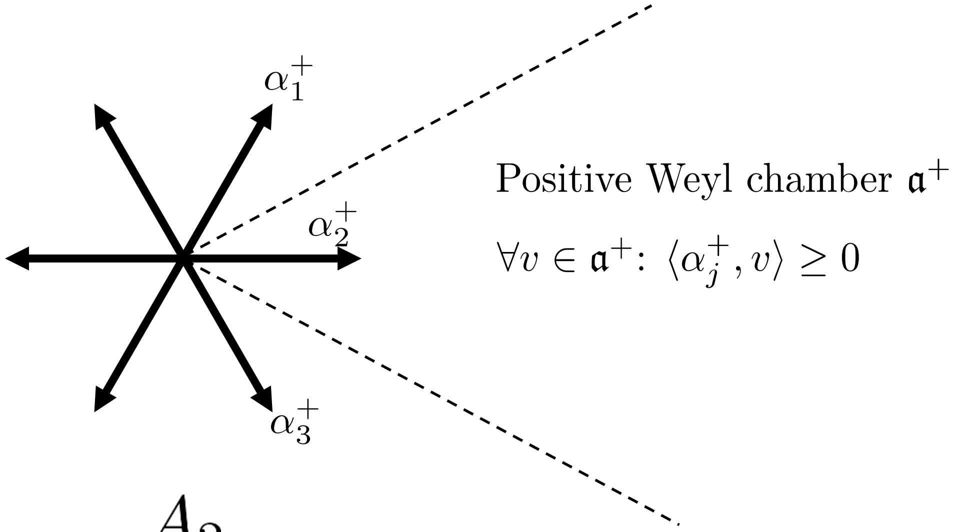
Cartan subspace $\mathfrak{a}$: maximal connected, totally geodesic, flat sub-manifold of $\mathbb{X}$

$$\mathfrak{a} \approx \mathbb{R}^{\ell} \qquad \text{and} \qquad {\color{red}\ell} = \dim \mathfrak{a} = \operatorname{rank} G/K$$

A_1



$A_1 \times A_1$



A_2

Weyl Chamber

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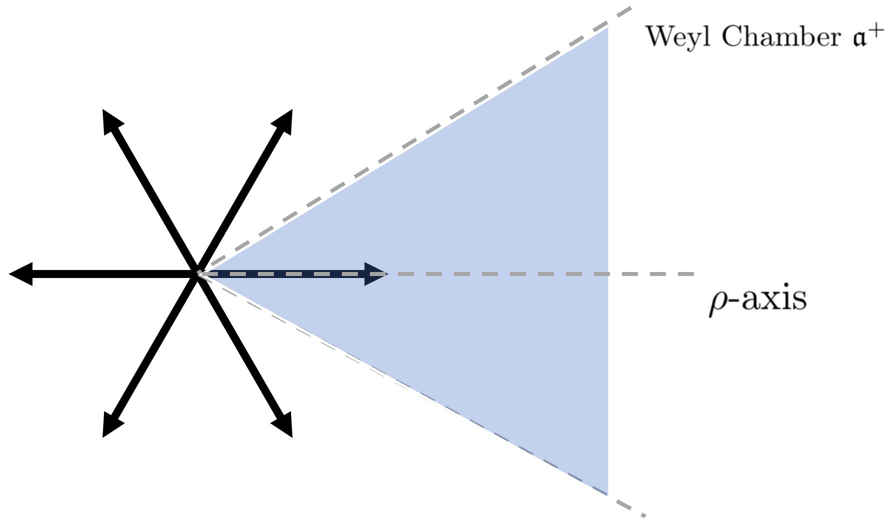
- Cartan decomposition $G = K(\exp \overline{\mathfrak{a}^{+}})K$



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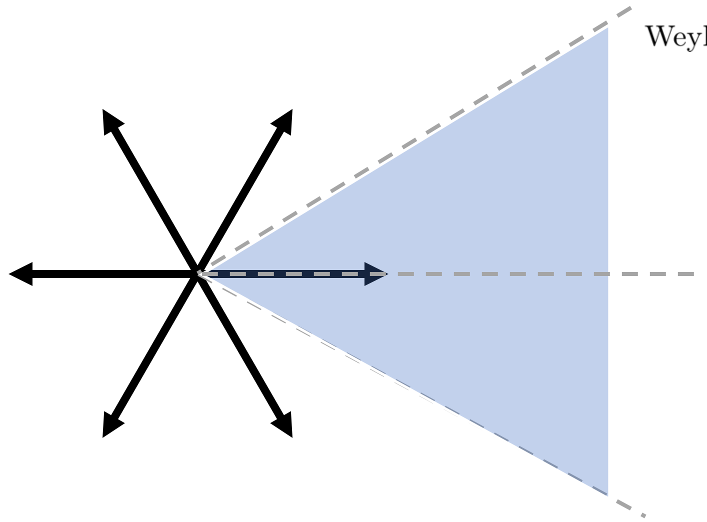
- Cartan projection $\mu : G \longrightarrow \overline{\mathfrak{a}^+}$ such that

$$g \in K e^{\mu(g)} K$$

- $d(e, \gamma e) = \|\mu(\gamma)\| \qquad \forall \gamma \in \Gamma$



In higer rank: $\rho \in \mathfrak{a}^+$ is a vector, known as $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$

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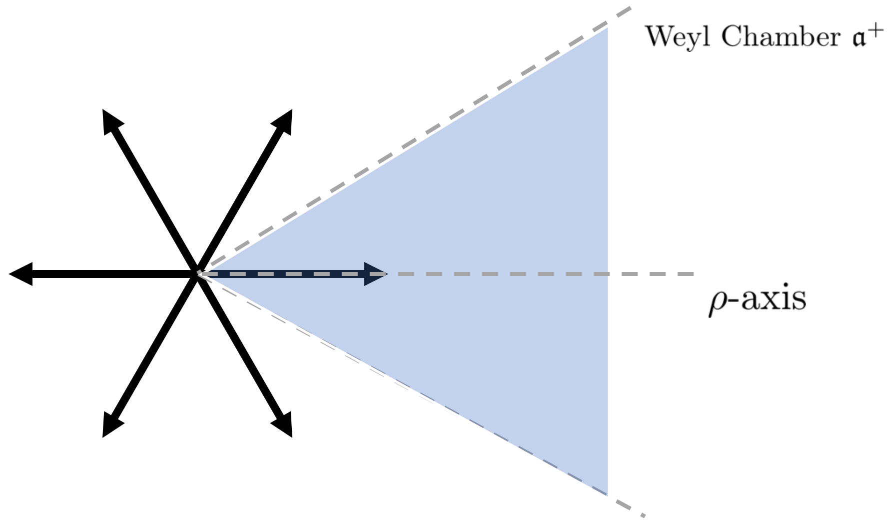

In higer rank: $\rho \in \mathfrak{a}^{+}$ is a vector, known as $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^{+}} m_{\alpha} \alpha$

General Characterization

$$\lambda_0(\Gamma\backslash X) = \|\rho\|^2$$

$$\updownarrow$$

$$\tilde{\delta}(\Gamma) \leq \|\rho\|$$

Theorem [Anker-Z. '22 *Geom. Dedicata*]

$$\lambda_0(\Gamma\backslash X) = \begin{cases} \|\rho\|^2 & \text{if } 0 \leq \tilde{\delta}_\Gamma \leq \|\rho\| \\ \|\rho\|^2 - (\tilde{\delta}_\Gamma - \|\rho\|)^2 & \text{if } \|\rho\| \leq \tilde{\delta}_\Gamma \leq 2\|\rho\| \end{cases}$$

Critical Exponent

- [Leuzinger '03]: Lower and upper bounds of $\lambda_0(\Gamma\backslash X)$ in terms of

$$\delta_\Gamma = \inf\left\{ s \in \mathbb{R} \,\middle|\, \sum_{\gamma\in\Gamma} e^{-s\|\mu(\gamma)\|} < \infty \right\}$$

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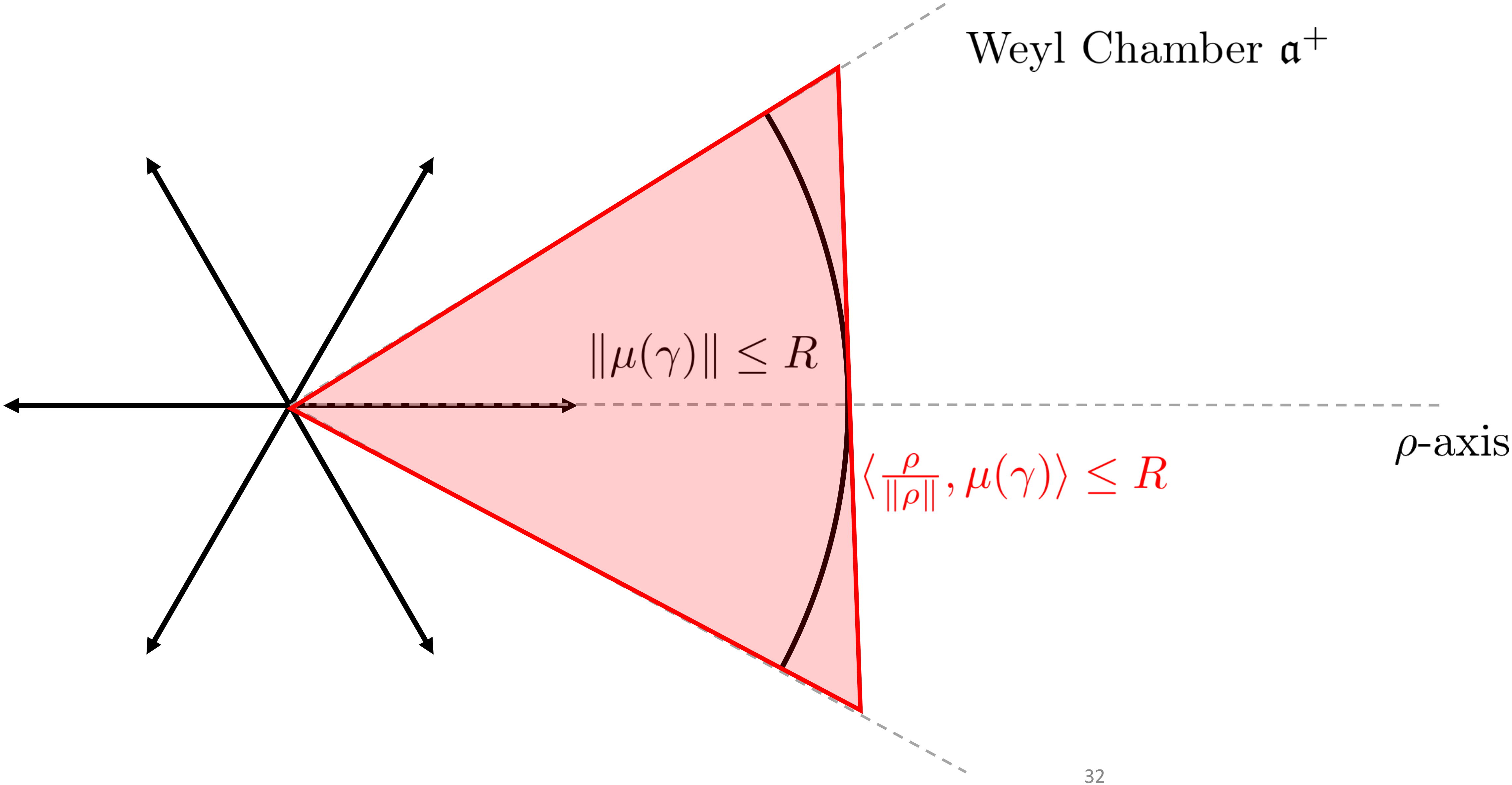
- [Carron-Pedon '04, Anker-Z. '22]: Introduce the modified critical exponent

$$\tilde{\delta}_\Gamma = \inf\left\{ s \in \mathbb{R} \,\middle|\, \sum_{\gamma\in\Gamma} e^{-\min\{s,\|\rho\|\}\langle \frac{\rho}{\|\rho\|},\mu(\gamma)\rangle - \max\{0,s-\|\rho\|\}\|\mu(\gamma)\|} < \infty \right\}$$

- $0 \leq \delta_\Gamma \leq \tilde{\delta}_\Gamma \leq 2\|\rho\|$ and $\delta_\Gamma = \tilde{\delta}_\Gamma$ in rank 1

Critical Exponent



Temperedness

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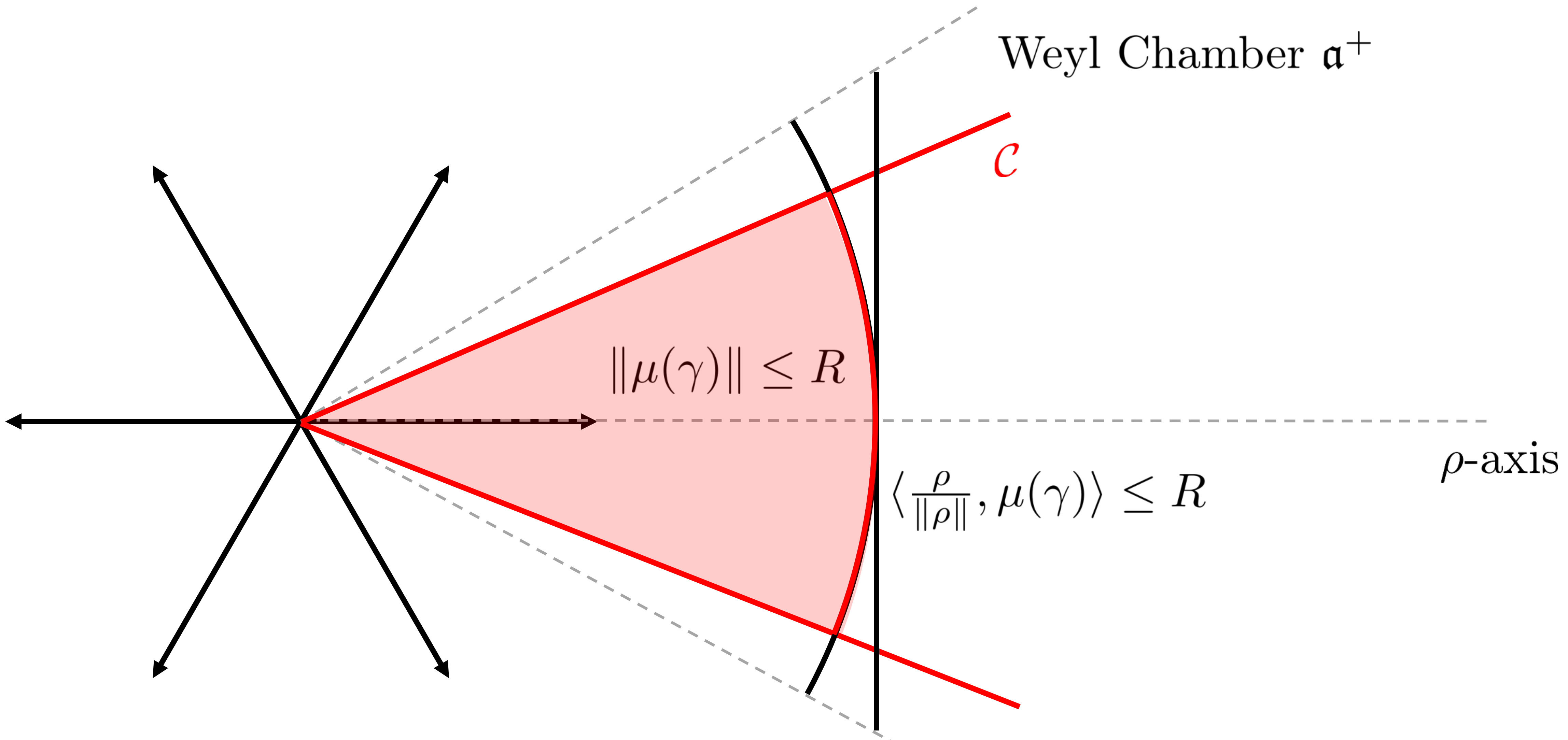
\lambda_0(\Gamma\backslash X) = \|\rho\|^2 \arrow[d, leftrightarrow] & L^2(\Gamma\backslash G) \text{ is tempered} \arrow[d, leftrightarrow] \\
\tilde{\delta}(\Gamma) \leq \|\rho\| & \psi_\Gamma(H) \leq \rho(H) \quad \forall H \in \mathfrak{a}
\end{tikzcd}
$$

Equivalence

- [Edwards-Oh '23 *Commun. Am. Math. Soc.*] : if Γ is Anosov
- [Lutsko-Weich-Wolf '24] : in general

Growth Indicator Function



$\psi_\Gamma(H) = \|H\| \inf_{H\in\mathcal{C}} \inf\{s \in \mathbb{R} \mid \sum_{\gamma\in\Gamma,\mu(\gamma)\in\mathcal{C}} e^{-s\|\mu(\gamma)\|} < \infty\}$ [Quint '02 *GAFA*]

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$$

Theorem [Wolf-Z. '24 *PAMS*]

$$
\tilde{\delta}_\Gamma = \begin{cases} \sup_{H \in \overline{\mathfrak{a}^+}} \psi_\Gamma(H) \cdot \frac{\|\rho\|}{\rho(H)} & \text{if} \quad \psi_\Gamma \leq \rho \\ \sup_{H \in \overline{\mathfrak{a}^+}} \frac{\psi_\Gamma(H) - \rho(H)}{\|H\|} + \|\rho\| & \text{otherwise} \end{cases}
$$

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\end{tikzcd}
$$

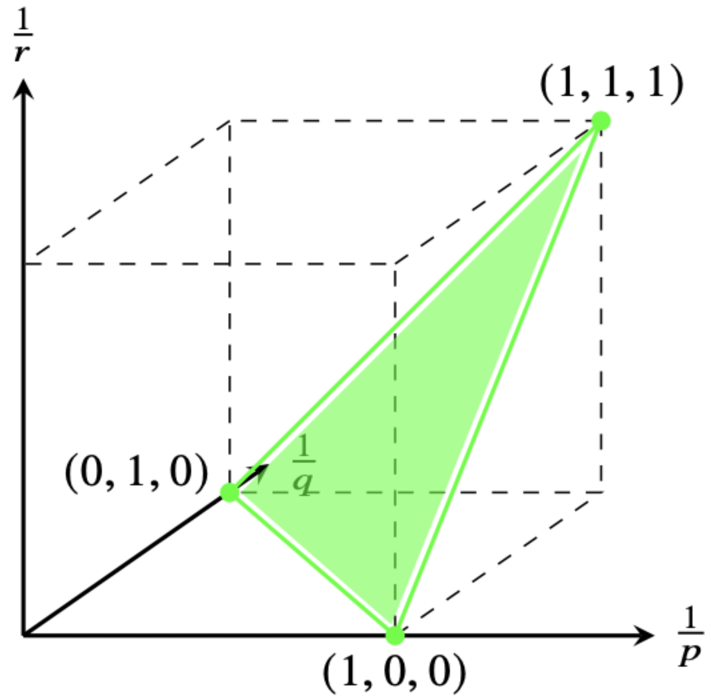
Theorem [Wolf-Z. '24 *PAMS*]

$$\lambda_0(\Gamma\backslash X) = \|\rho\|^2 - \max\left\{0, \sup_{H \in \overline{\mathfrak{a}_+}} \frac{\psi_\Gamma(H) - \rho(H)}{\|H\|}\right\}^2$$

Kunze-Stein Phenomenon

- Young's inequality: $L^p(G) * L^q(G) \subset L^r(G)$ $\qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$

$$L^{1}(G) * L^2(G) \subset L^2(G)$$



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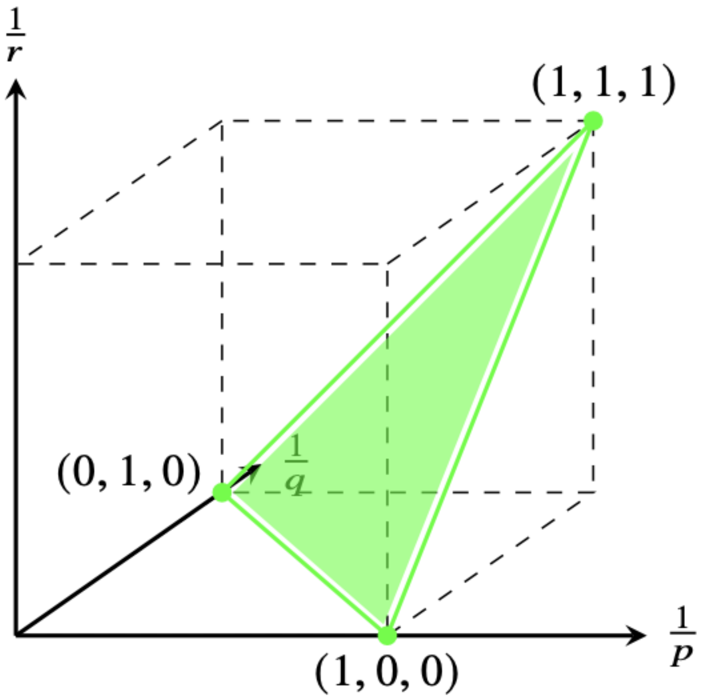
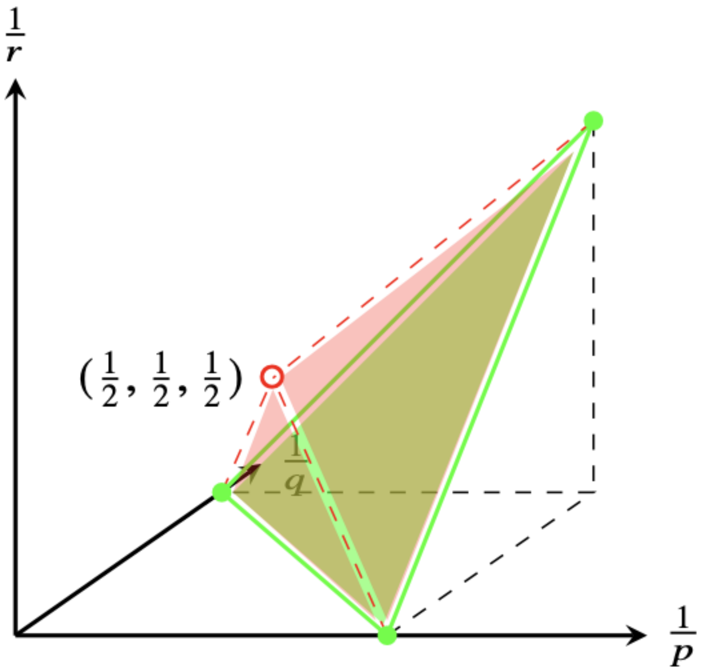
$$L^1(G) * L^2(G) \subset L^2(G)$$

- Kunze-Stein: $L^p(G) * L^2(G) \subset L^2(G)$ $\qquad 1 \le p < 2$





PDE Motivation

e.g. Free Schrödinger equation: $(i\partial_t + \Delta_x)u(t,x) = 0, \quad u(0,x) = f(x)$

whose solution is given by $u(t,x) = e^{it\Delta} f(x)$

Strichartz estimate

$$\|u\|_{L_t^p(\mathcal{I}, L_x^q(\mathcal{M}))} = \left(\int_{\mathcal{I}} \mathrm{d}t \, \|u\|_{L^q(\mathcal{M})}^p \right)^{1/p} \lesssim \|f\|_{H^s(\mathcal{M})}$$

- for all **admissible pairs** (p,q)
- $s = 0$: without loss; $s > 0$: with loss of derivatives
- $\mathcal{I}$ bounded: local-in-time; $\mathcal{I}$ unbounded: global-in-time

Strichartz Estiamte

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whose solution is given by $u(t,x) = e^{it\Delta} f(x)$

In $\mathbb{R}^n$ [..., Strichartz '77 *Duke*, ..., Keel-Tao '98 *AJM*]

Global-in-time Strichartz inequality **without loss**

$$\|u\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$$

holds for all **admissible** pairs (p,q), i.e.,

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} \qquad p \geq 2 \qquad (p,q) \neq (2,\infty)$$

Strichartz on manifolds

In $\mathbb{R}^n$

Global-in-time Strichartz holds without loss of derivatives

Compact manifold (M, g)

$$\|e^{it\Delta_g}u_0\|_{L^p_t(I, L^q_x(M))} = \left(\int_I \mathrm{d}t \, \|e^{it\Delta_g}u_0\|^p_{L^q(M)}\right)^{1/p} \lesssim \|u_0\|_{H^s(M)}$$

- I is bounded

Strichartz on manifolds

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- I is bounded
- $\mathbb{T}^n$ [Bourgain '93 *GAFA*]: $s > \frac{n}{4} - \frac{1}{2}$
- M [Burq-Gérard-Tzvetkov '04 *AJM*]: $s = \frac{1}{p}$

Strichartz on manifolds

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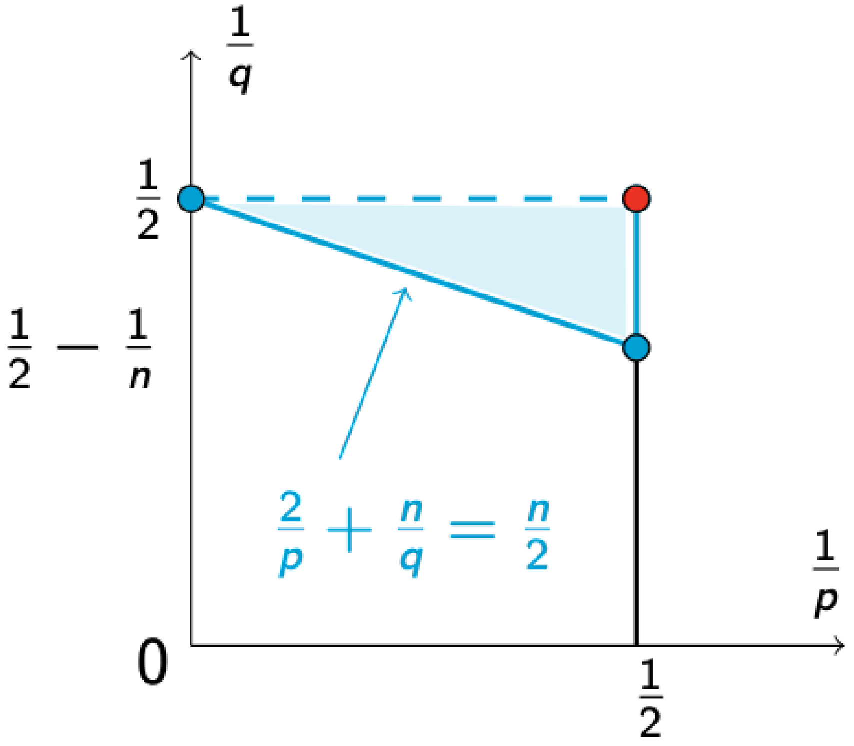
Question: on which manifolds does Strichartz hold without any loss ?

Strichartz on G/K

Global-in-time Strichartz estimate

$$\|e^{it\Delta}u_0\|_{L^p_t(\mathbb{R},L^q_x(G/K))} \lesssim \|u_0\|_{L^2(G/K)}$$

holds without any loss of derivatives for all (p,q) admissible:



[Anker-Meda-Perfelice-Vallarino-Z. '23 *JDE,* Anker-Z. '24 *AJM*]

On Locally Symmetric Space

Strichartz estimate

Global-in-time Strichartz inequality holds without lossing any derivatives for the large X-admissible set if the following conditions are met:

- X has rank 1
- Γ is convex cocompact
- $\delta_\Gamma < \rho$

[Burq-Guillarmou-Hassell '10 *GAFA*, Fotiadis-Mandouvalos-Marias '18 *Math. Ann.*]

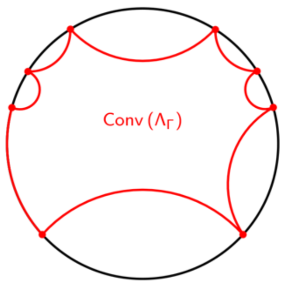


Figure: Convex hull of the limit set Λ_Γ in $\mathbb{H}^2$

$$\Gamma \text{ convex cocompact} \iff \Gamma\backslash\mathrm{Conv}(\Lambda_\Gamma) \text{ compact}$$

On Locally Symmetric Space

Strichartz estimate

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Remark.

- δ_Γ small enough $\Longrightarrow$ Γ is convex cocompact [Liu-Wang '23 *GT*]
- $\delta_\Gamma < \rho$ $\Longrightarrow$ temperedness $\Longrightarrow$ Kunze-Stein [Z. '20 *JGA*]