Scalar curvature and volume entropy of hyperbolic 3-manifolds

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Thm (Kazaras - Song - X. '23) $\forall$ closed hyperbolic 3-mfd M, there $\exists$ metric g

s.t. ① $R_g \geq -6$

② $h(g) > 2$ strictly.

Rmk. 1. $\boxed{R_g}$ = scalar curvature of g.

= "average of sectional curv on all directions"

Fact: $|B(x,r)| = |B^n(1)| \cdot r^n - \frac{R_g(x)}{6(n+2)} |B^n(1)| r^{n+2} + O(r^{n+4})$. for $\boxed{r \ll 1.}$

$r \ll 1$

Warning: $R \geq 0 \not\Rightarrow$ volume comparison.

($Ric \geq 0 \Rightarrow |B(x,r)| \leq |B^n(1)| r^n$.)

(Schoen-Yau, Gromov-Lawson, Stern.)

Thms. ① (Geroch Conjecture) T^n has no $R > 0$ metrics.

② (Aspherical Conj) $n \leq 5$, M^n closed aspherical $\Rightarrow$ no $R > 0$ metrics.

(Chodosh-Li '24) $K(\pi, 1)$

(n ≥ 6 open)

Conj (Gromov). M^n closed, $R \geq R_0 > 0$. Then "M is large in $\leq (n-2)$ direction"

Model: $M = N^{n-2} \times S^2(\varepsilon)$ $R > 0$ when $\varepsilon \ll 1$.

2. $h(g)$ = volume entropy of g.

$:= \lim_{r \to \infty} \frac{1}{r} \log |\tilde{B}(x_0, r)|$

geodesic ball in $\tilde{M}$

$\longrightarrow$ If $|\tilde{B}(x_0, r)| \approx A e^{\boxed{B} r}$

then $B = h(g)$

From Bishop-Gromov $\Rightarrow$ $h(g)$ exists.

If $Ric \geq -(n-1)$ then $h(g) \leq (n-1)$.

"=": M = closed hyperbolic mfd.

Thm (Kazaras - Song - X. '23) $\forall$ closed $\boxed{\text{hyperbolic}}$ 3-mfd M, there $\exists$ metric g

s.t. ① $R_g \geq -6$

② $h(g) > 2$ strictly.

Relations. M: closed hyp. 3-mfd. g_0 = hyp. metric.

g = any metric

Thm:

$$\begin{tikzcd}
R_g \geq -6 \arrow[dr, "\text{Ricci flow with surgery (Perelman)}"] \arrow[d, "?"'] & \\
h(g) \leq 2 \arrow[r, "\text{Besson-Courtois-Gallot '95}"'] & |M|_g \geq |M|_{g_0}
\end{tikzcd}$$

Note: $n \geq 4$ case is open (Schoen's conj).

Conj (Agol - Storm - Thurston '07) Does $R_g \geq -6 \Rightarrow h(g) \leq 2$? (Main thm: not the case)

Thm:

$$\begin{tikzcd}
R_g \geq -6 \arrow[dr, "\text{Munteanu-Wang, Davaux}"] \arrow[d, "?"'] & \\
h(g) \leq 2 \arrow[r] & \lambda_1(-\tilde{\Delta}) \leq 1
\end{tikzcd}$$

Δ on the $\tilde{M}$.

$\Uparrow$

$\frac{1}{4} h(g)^2 \geq \lambda_1(-\tilde{\Delta})$ by direct verification.

Fact: $\boxed{R_g \geq -6 \Rightarrow Ch(\tilde{M}) \leq 2}$

$Ch(\tilde{M}) = \inf \left\{ \frac{|\partial E|}{|E|} : E \subset\subset \tilde{M} \right\}$

Fact (Cheeger's ineq)

+

$\boxed{\lambda_1(-\tilde{\Delta}) \geq \frac{1}{4} Ch(\tilde{M})^2}$

Summary, heuristics (!): $\boxed{\text{Scalar curvature}}$ $\overset{✓}{\longleftrightarrow}$ $\boxed{\text{volume, area (codimension 1)}}$

$\boxed{\text{Scalar curvature}}$ $\overset{?}{\longleftrightarrow}$ $\boxed{\text{distance function.}}$

Another evidence: "Schoen - Yau min. surf. technique"

Limit space. $\sec \geq -k \longrightarrow$ "Alexandrov space", in terms of distance ; d_{GH}

$\text{Ric} \geq -k \longrightarrow$ "RCD space", distance + "volume measure" ; d_{mGH}

$R \geq -k . \longrightarrow$? (open, unknown)

Sormani: "$\boxed{\text{integral current space}}$", $\boxed{\text{intrinsit flat}}$ convergence (?)

heuristic: "volume convergence"

underlying distance.

Thm (Sormani–Wenger) If $X \overset{IF}{\cong} Y$ then $X \overset{iso}{\cong} Y$.

Thm (Kazaras–X '25) Suppose $n \geq 3$, M^n oriented, $\Sigma^{n-2} \subset M$ closed oriented.

$v_0 \ll -1$ constant

Then $\forall \varepsilon > 0$ and $\boxed{v_0 \in C^\infty(\Sigma),\ v_0 \leq 0}$, there $\exists$ metric g' such that:

much more computation (!)

(1) $g' = g$ outside $N_g(\Sigma, \varepsilon)$

$\ll 1$ const

(2) $\boxed{g'|_\Sigma = \boxed{e^{2v_0}}\, g|_\Sigma}$ length$(\Sigma) \ll 1$

Goal: contract the diam of Σ.

(3) $R_{g'} \geq R_g - \varepsilon$ pointwise,

(4) $|N_g(\Sigma, \varepsilon)|_{g'} \leq 12\pi\, |\Sigma_g| \cdot \varepsilon^2$

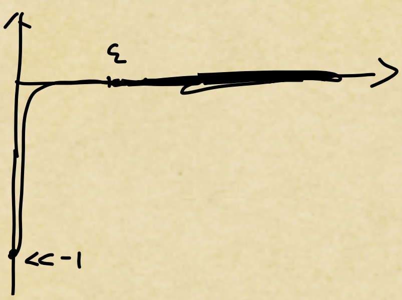
(5) $\forall x \in \partial N_g(\Sigma, \varepsilon)$ it holds $d_{g'}(x, \Sigma) \leq 3\varepsilon$

"Pf": $\boxed{\text{assume } M^3 \text{ hyp}, \ \Sigma = \text{closed geodesic}}$, $v_0 \ll -1$ const.

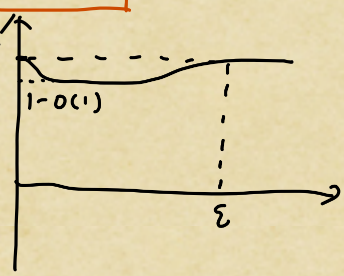
Near Σ, $g = \underbrace{g_0}_{\text{hyp metric}} = dr^2 + sh(r)^2 d\theta^2 + ch(r)^2 dt^2$

Now set $g' = \boxed{e^{-2u} dr^2 + e^{-2u} h^2 sh(r)^2 d\theta^2} + \boxed{e^{2u} ch(r)^2 dt^2}$

where $u =$



, $h =$



KSX'23

picture:

$g' =$

$+$ ... dt^2

Rmk: 1. the name is "drawstring"

"drawstring bag"

2. inspired by Lee-Naber-Neumayer (created drawstring in $\Sigma^1 \subset T^n$, $n \geq 4$)

closed geodesic

flat torus

codim ≥ 3

KX25: $\boxed{\text{codim } 2}$ (more difficult: 2D cone is flat

3D cone has $R \geq O(\frac{1}{r^2})$)

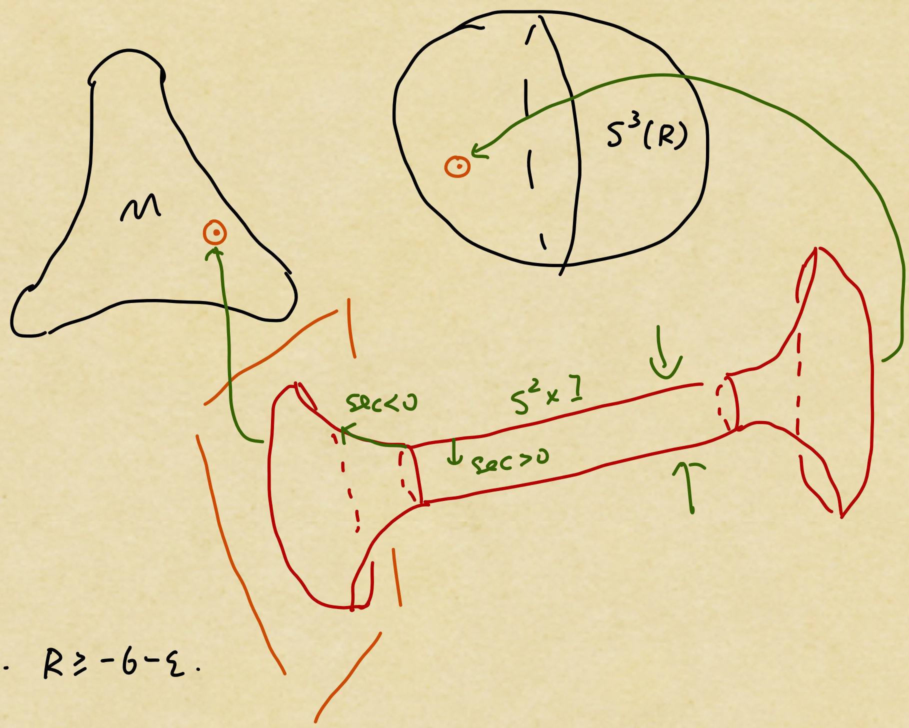
Thm (Kazaras-Song-X. '23) $\forall$ closed hyperbolic 3-mfd M, there $\exists$ metric g

s.t. ① $R_g \geq -6$

② $h(g)$ arbitrarily large.

Attempt: tunnel construction (Schoen-Yau, Gromov-Lawson)

(M^3, g_0) hyp, closed.



Fact: we can manage s.t. $R \geq -6-\varepsilon$.

$\rightsquigarrow$ metric on $M \# S^3 = M$

Let $R \gg 1$. $\rightsquigarrow$ M has arbitrarily large volume.

Q: does this increase $h(g)$?

A: No. In $\tilde{M}$, let $N(r) = \#\{$fundamental domain that intersect $\tilde{B}(x_0, r)\}$.

Then $|\hat{B}(x_0,r)| \simeq |M| \cdot N(r)$.

$$r^{-1}\log|\hat{B}(x_0,r)| \simeq \frac{\log N(r)}{r} + \frac{\log|M|}{r}$$

Summary: Shorten the distance between points.

Pf of main thm:

Let γ be closed geodesic. Create drawstring around γ.

Lemma (Balancheff - Merlin) Suppose $\gamma_1, \gamma_2 \subset (M,g)$ passing through x, γ_1, γ_2 generate rk 2 free group.

Then
$$\frac{1}{1+e^{h(g)|\gamma_1|}} + \frac{1}{1+e^{h(g)|\gamma_2|}} \leq \frac{1}{2}.$$

$\Downarrow$

If $|\gamma_1| \ll 1$, $|\gamma_2| \leq C$ then $h(g) \gg 1$.

$a \geq 1$ $a \in \mathbb{N}$.

Apply Lemma with $\gamma_1 = a\gamma$

γ_2 = chosen fixed curve that is $\perp \gamma_1$