Seiberg-Witten equations on end-periodic 4-mfd and psc metric.

Q: Which 4-mfds admits psc metric?
closed.

- index theory for Dirac operator

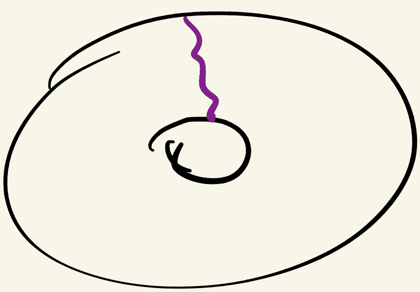
$$\Rightarrow X \text{ psc}, \text{ spin} \quad \sigma(X) = 0$$

$$I_X := H^2(X;\mathbb{R}) \otimes H^2(X;\mathbb{R}) \to \mathbb{R}$$
$$\alpha \otimes \beta \longmapsto \langle \alpha \cup \beta, [X] \rangle$$

$$\sigma(X) = \sigma(I_X)$$

- minimal surface.

$b_1(X) > 0$ can find minimal hypersurface with psc metric.



$S^2 \times S^1$, S^3, S^3/G
connected sum.

- Seiberg–Witten invariant.

$(X, \mathfrak{s})$

$\uparrow$ spin^c structure

$$\begin{array}{c} S^1 \\ \downarrow \\ \mathrm{Spin}^c(4) = \{(A,B) \in U(2)\times U(2) \mid \det A = \det B\} \\ \downarrow \\ SO(4) \end{array}$$

$$SO(4) \hookrightarrow Fr \to X \overset{\mathfrak{s}}{\rightsquigarrow} \mathrm{Spin}^c(4) \hookrightarrow P \to X$$

$$\Downarrow$$

$$\Lambda^* T^* X \xrightarrow{\rho} \mathrm{Hom}(S, S) \qquad \mathbb{C}^2 \hookrightarrow S^{\pm} \to X$$

$$S = S^+ \oplus S^-$$

Seiberg–Witten equation

$$\begin{cases} F_{A^t}^+ = \rho^{-1}(\phi\phi^*)_0 + \underline{\underline{\omega}} \\ \not{D}_A^+ \phi = 0 \end{cases}$$

$\uparrow$ perturbation

A : connection on P lifts Levi-Civita conn. on Fr.

$\phi \in \Gamma(S^+)$

A^t : induced connection on $\det(S^+)$

Witten : under psc metric

Seiberg–Witten eq. has no irreducible solution $(\phi \neq 0)$

$b_2^+(X)>1$, $\underset{\mathbb{Z}}{SW(X,\mathfrak{s})} \in \mathbb{Z}$ $:= \#$ irreducible solution of SW $/$ gauge.

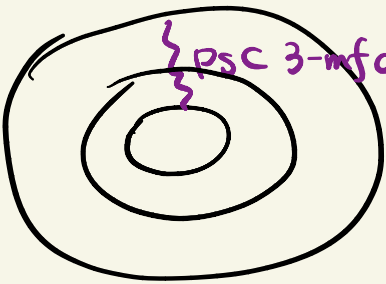
If X has psc metric $\Rightarrow SW(X,\mathfrak{s})=0$

E.g. Taubes if X symplectic $b^+(X)>1$

$\Rightarrow X$ has no psc metric.

- $X \quad H_*(X) \cong H_*(S^1 \times S^3)$

  $b_1(X)=1 \quad b_2(X)=0 \qquad$ psc?

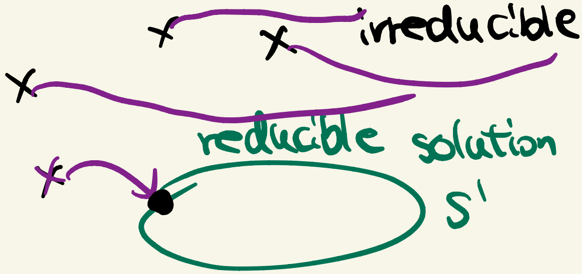


Seiberg–Witten eq.

$SW(X,\mathfrak{s})$ <u>not well-defined</u>

To define Seiberg-Witten eq. pick (g_0, ω_0) (ω_0: perturbation 2-form)

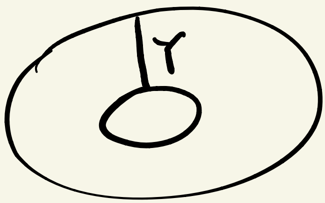
$$\begin{array}{c} (g_0,\omega_0) \\ \downarrow \; \underline{(g_t,\omega_t)} \\ (g_1,\omega_1) \end{array}$$



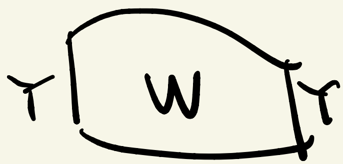
$\exists$ reducible solution $(A,0)$ s.t. $\operatorname{Ker} \not{D}_A \neq 0$

Mrowko – Ruberman – Saveliev :

$$\underbrace{\lambda_{SW}(X)}_{\text{independent with } (g,\omega).} = \underbrace{\# \mathcal{M}(g,\omega)}_{\text{solution/gauge}} + \underbrace{n(g,\omega)}_{\text{index correction term}}$$



$b_1(X) = 1$

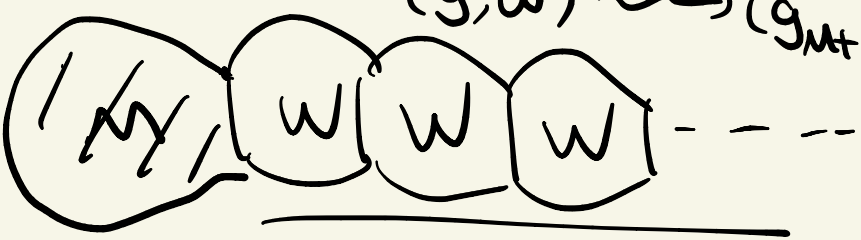


end-periodic manifold.

extend over M

$(\tilde{g}, \tilde{\omega}) \rightsquigarrow (\tilde{g}_{M_+}, \tilde{\omega}_{M_+})$

$\partial M = Y$

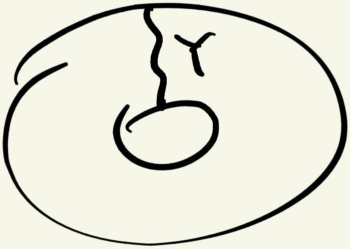


$$M_+ = \underset{\partial M = Y}{M} \cup W \cup W \cup W \cup W \cdots$$

$$\mathrm{ind}_{\mathbb{C}}(\not{D}(M_+, \tilde{g}_{M_+}, \tilde{\omega}_{M_+})) + \frac{\sigma(M)}{8}$$

$$:= n(g, \omega) \in \mathbb{Z}$$

$$\simeq -\frac{\sigma(WU \cdots)}{8}$$



another invariant of X

$\exists$ 3-mfd $Y \subset X$

s.t. $[Y] = 1 \in H_3(X)$

$b_1(Y) = 0$

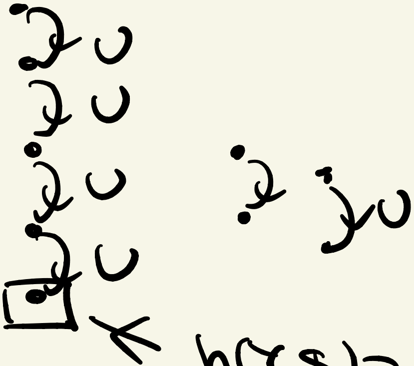
$$\boxed{h(Y) = \text{Froyshov invariant of } Y.}$$

Y : 3-mfd $\quad Y \times \mathbb{R}$ Seiberg-equation

monopole Floer homology $\quad \deg(U) = -2$

$\widecheck{HM}(Y, \mathfrak{s})$: module over $\mathbb{Z}[U]$

$\cong \mathbb{Z}[U, U^{-1}] / (U^{-1}) \oplus$ torsion



$h(Y,\mathfrak{s}) =$ deg of bottom of U-tower.

$h(X) := h(Y)$

Thm: If X is psc

$$\lambda_{SW}(X) + h(X) = 0$$

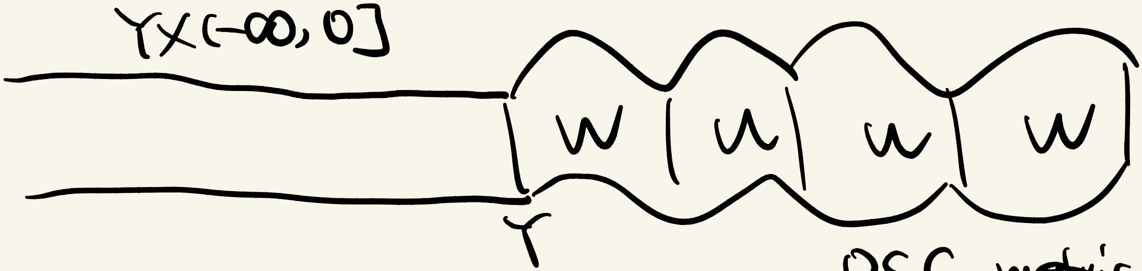
Corollary: If $H_3(X)$ is generated by

Y with $h(Y) \not\equiv \underset{\text{Rokhlin invariant.}}{\underbrace{\rho(Y)}}$ mod 2

Then X has no psc.

(E.g. $Y = \Sigma(2,3,7)$)

Sketch of proof



PSC metric

Seiberg-Witten eq.

monopole Floer $\quad -\frac{\sigma}{8} = n(\underline{\underline{g}}, \omega) = \lambda_{SW}$

Y | "compact" 4-mfd

$\Rightarrow h(Y) = -\lambda_{SW}.$ $\quad$ [Ch]

$\boxed{\text{Mazur}} \simeq *$

exotic $\mathbb{R}^4$