Euler characteristic and simplicial volume of closed nonpositively curved four-manifolds

— joint with Inkang Kim.

- Let M be a closed, connected, oriented n-dim. mfd.

simplicial volume (Gromov, 1982 IHES):

$$\|M\| = \inf \{ \textstyle\sum_i |a_i| : [\sum_i a_i \sigma_i] = [M] \in H_n(M, \mathbb{R}) \}$$

where the infimum take all real singular cycles representing $[M]$.

- Question: when $\|M\| > 0$?
- For example, $M = S^1$, $\|M\| = 0$ $\quad$ $\inf\{\frac{1}{n}\sigma_i\} = 0$
- hyperbolic mfds: $\|M\| = c\,\mathrm{Vol}(M)$ (>0) for some $c>0$.
  by. Gromov. (1982') and Thurston's book (The geometry and topology of three-mfd.)
- Strictly negative sectional cur. $\|M\|>0$.
  by Gromov (82') and Inoue-Yano (82')
- Closed locally symmetric spaces of non-cpt type
  $\|M\|>0$ (Lafont-Schmidt 06' Acta)
- Gromov's conj 1: M, nonpositive sectional cur. and $\mathrm{Ric}<0$
  then $\|M\|>0$.

Regarding this conj: Connell-Wang (20' Math Ann.). $\|M\|>0$ for dim$=3$.

Additionally, close relationship between $\chi(M)$ and $\|M\|$

For instance: M. supports an affine flat bundle E of same dim.

$\|M\| \geq |\chi(E)|$

Gromov conj (93') 2: If M is aspherical, $\|M\|=0$, then $\|M\|=0$.

($\tilde{M}$ is contractible).

Recently. for closed nonpositively curved 4-mfds.

Connell-Ruan-Wang. show Conj 2 $\Rightarrow$ conj 1.

by observing that. $Ric = 0$ at some point if $\chi(M) = 0$.

and proposed the following conjecture.

- Connell-Ruan-Wang's conj 3: Let M be a closed nonpositively curved 4-mfd. Then $\|M\| = 0 \Leftrightarrow \chi(M) = 0$.

For M is real analytic nonpositively curved 4-mfd. Connel-Ruan-Wang proved: $\chi(M) = 0 \Rightarrow \|M\| = 0$.

- Using G-B Thm due to Allendoerfer-Weil. we prove:

Thm: Let M be a closed nonpositively curved 4-mfd. Then

$$\|M\| \geq \frac{1}{11}|\chi(M)|$$

In particular. if $\chi(M) \neq 0$ then $\|M\| > 0$.

Cor: If M. $\dim M = 4$. nonpositive sectional cur. $Ric < 0$.

then $\|M\| > 0$.

- Geodesic simplices: Let (M, g) be a closed Riemannian mfd with nonpositive sectional cur, $p: \tilde{M} \to M$ $(\tilde{M} \cong \mathbb{R}^n)$.

$(\tilde{M}, p^*g)$. The standard simplex

$$\Delta^k = \{(x_1 \cdots x_{k+1}) \in \mathbb{R}^{k+1} : \sum_{i=1}^{k+1} x_i = 1,\ x_i \geq 0\}$$

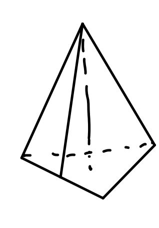
and identify Δ^{k-1} with $\{(x_1 \cdots x_{k+1}) \in \Delta^k,\ x_{k+1} = 0\}$

Define geodesic k-simplex inductively:

- $k=0$ $\sigma_{p_0}: \{1\} \to \{p_0\} \subset \tilde{M}$
- Assuming $\sigma_{p_0 \cdots p_{k-1}}$ is defined, $\sigma_{p_0 \cdots p_k}: \Delta^k \to \tilde{M}$ by

$$\sigma_{p_0 \cdots p_k}((1-t)s + t(0 \cdots 0\ 1)) = \gamma(t)$$

for each $s \in \Delta^{k-1} \subset \Delta^k$. $\gamma(t)$ is the unique geodesic. each geodesic k-simplex is a smooth singular simplex.



$\sigma : \Delta^k \to M$, $\quad str(\sigma) = p \circ \hat{\tilde{\sigma}}$

$\hat{\tilde{\sigma}}$: geodesic k-simplex with same vertices as a lift of σ.

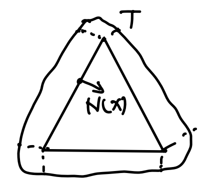
$$\|M\| = \inf \{ \sum_i |a_i| , \ [\sum a_i \sigma_i] = [M], \ \sigma_i = p \circ \hat{\tilde{\sigma}}_i \}.$$

- Gauss-Bonnet thm for Riem. simplices.

Let M be a simplex equipped with a smooth metric.

$M[r]$: r-dim faces of ∂M, $\quad M[n] = M \setminus \partial M$.

$\forall x \in M[r]$, $\quad N(x) \subset S(x)$, that point inward toward $M[n]$.



$\pi : T \to M$

$M \hookrightarrow \mathbb{R}^N \qquad G : T \to S^N \qquad T[r] = \pi^{-1}(M[r])$

The contributions to the degree of the Gauss map are given by

$$G(M[n]) = \frac{1}{\omega_N} \int_{T[n]} G^*(d\xi) = \int_{M[n]} \Phi_n(x)\, dv(x).$$

volume of S^N

$$G(M[r]) = \frac{1}{\omega_N} \int_{T[r]} G^*(d\xi) = \int_{M[r]} dv(x) \int_{N(x)^*} \Phi_r(x, \xi)\, d\xi.$$

ξ is unit inward normal vector.

$N(x)^* = \{ \xi \in T_x(M[r]) \mid \langle \xi, N(x) \rangle \geq 0, \ |\xi| = 1 \}.$

G-B thm :

$$1 = G(M[n]) + G(M[n-1]) + \cdots + G(M[0]).$$

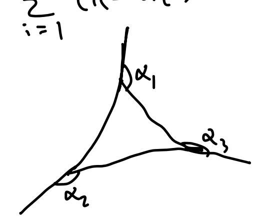
Ex: $\quad n = 2$.

$$\frac{1}{2\pi} \int_{M[2]} K\, dv(x) + \frac{1}{2\pi} \int_{M[1]} dv(x) \int_{N(x)^*} \frac{1}{r} \Lambda_{11}(\xi)\, d\xi$$

$$+ \frac{1}{2\pi} \sum_{i=1}^{3} \mathrm{Vol}\,(N(x_i)^*) = 1.$$

For geodesic simplex : $\quad \Lambda_{11}(\xi) = 0$.

$$\int_{M[2]} K\, dv(x) = 2\pi - (\alpha_1 + \alpha_2 + \alpha_3) = -\pi + \sum_{i=1}^{3} (\pi - \alpha_i) \geq -\pi.$$



$\alpha_i = \mathrm{Vol}\,(N(x_i)^*)$, exterior angle :

Proof of Main thm:

Let $\tilde\sigma = \sum_{i=1}^{N} a_i \tilde\sigma_i$ be a chain in $\tilde M$, $[p\circ\tilde\sigma] = [M]$.

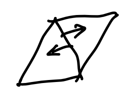
Let $\sum_{i=1}^{N_0} a_i \tilde\sigma_i$ be non-degenerate part. $\dim \tilde\sigma_i(\Delta_n) = n$.

$\sigma_i = p\circ\tilde\sigma_i$ $\quad N_0 \le N$.

then $\left[\sum_{i=1}^{N_0} a_i \sigma_i\right] = [M]$.

$$\chi(M) = \int_M \Psi_4(x)\,d\nu(x) = \sum_{i=1}^{N_0} a_i \int_{\tilde\sigma_i} \tilde\Psi_4(\tilde x)\,d\nu(\tilde x).$$

since $\sum_{i=1}^{N_0} a_i \int_{\partial\sigma_i} w = 0$, and $-\Psi_3(x,\xi) = \Psi_3(x,-\xi)$



we obtain:

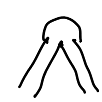
$$\sum_{i=1}^{N_0} a_i \int_{\partial\tilde\sigma_i} \tilde\Psi_3(\tilde x,\xi_0)\,d\nu(\tilde x) = 0$$

$$\chi(M) = \sum_{i=1}^{N_0} a_i\left[\int_{\tilde\sigma_i}\tilde\Psi_4(\tilde x)\,d\nu(\tilde x) + \int_{\partial\tilde\sigma_i}\tilde\Psi_3(\tilde x,\xi_0)\,d\nu(\tilde x)\right]$$

$$\underset{\text{G-B}}{=} -\sum_{i=1}^{N_0} a_i\left[\int_{\tilde\sigma_i[2]}\int_{N(\tilde x)^*}\tilde\Psi_2(\tilde x,\xi) + \int_{\tilde\sigma_i[0]}\int_{N(\tilde x)^*}\tilde\Psi_0(\tilde x,\xi)\right]$$

$$-\underbrace{\sum_{i=1}^{N_0} a_i\left[\int_{\tilde\sigma_i[1]}\int_{N(\tilde x)^*}\tilde\Psi_1(\tilde x,\xi)\right]}_{\substack{\shortparallel \\ 0 \text{ by geodesic simplex}}} + \sum_{i=1}^{N_0} a_i$$

$\tilde\Psi_1(\tilde x,\xi) = \frac{1}{2\pi^2\gamma}\Lambda_{11}(\xi)$



- $\int_{\tilde\sigma_i[0]}\int_{N(x)^*}\tilde\Psi_0(\tilde x,\xi) = \frac{1}{\omega_N}\int_{T[0]} g^*(d\xi) \le 5$ (five vertices)

- $\tilde\Psi_2(\tilde x,\xi) = \dfrac{R_{1212} + 2\det\Lambda(\xi)}{4\pi^2\gamma} = \dfrac{R_{1212} - 2\Lambda_{12}(\xi)^2}{4\pi^2\gamma} \le 0.$

$$\Rightarrow 0 \le \int_{N(\tilde x)^*}(-\tilde\Psi_2(\tilde x,\xi))\,d\xi \le \int_{S(\tilde x)}(-\Psi_2(x,\xi))\,d\xi = -\tilde\Psi_2(\tilde x) = -\frac{1}{2\pi}K_{\tilde\sigma_i[2]}(\tilde x)$$

For $\tilde\sigma_i$, it has $\binom{5}{3} = 10$ geodesic 2-simplices

$$\Rightarrow \left|\int_{\sigma_i[2]}\int_{N(\tilde x)^*}\tilde\Psi_2(\tilde x,\xi)\right| \le \left|\frac{1}{2\pi}\int_{\sigma_i[2]}K_{\tilde\sigma_i[2]}(\tilde x)\right| \le \frac12\binom{5}{3} = 5$$

$$\Rightarrow |\chi(M)| \le \left(\sum_{i=1}^{N_0}|a_i|\right)(1+5+5) = 11\sum_{i=1}^{N_0}|a_i| \le 11\left(\sum_{i=1}^{N}|a_i|\right)$$

taking infimum for all a_i. We obtain

$$\|M\| \ge \frac{1}{11}|\chi(M)|.$$