Positive Curvature Conditions and Contractible Manifolds

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Curvature and Topology

<u>**Classical Question**</u>: How does positive curvature control the topology of a a Riemannian manifold? Specifically,

I. Let M^{n+1} be an open manifold which supports a *complete* Riemannian metric with positive scalar curvature. Is M homeomorphic (or diffeomorphic) to the standard $\mathbb{R}^{n+1}$?

II. Let X^{n+1} be a compact manifold with boundary which supports a Riemannian metric with positive scalar curvature and mean convex boundary. Is X homeomorphic (or diffeomorphic) to the standard $(n+1)$-disk, $\mathbb{D}^{n+1}$?

Topological Background

First, recall that:

A. An *open* manifold is a non-compact manifold without boundary.

B. A *complete* Riemannian metric if all geodesics are defined for all time.

C. A manifold M is called *contractible* if it is homotopic to a point. Therefore, $\pi_i(M) = 0$, $i \geq 1$.

D. An open manifold M is called *tame* if it is the interior of compact manifold with boundary.

Conditions for Open Manifolds to be Diffeomorphic to $\mathbb{R}^n$

Dimension	Topology	Complete Metric	Curvature Condition	References
$n = 2$	open	yes	positive (scalar) curvature	Cohn-Vossen Huber
$n = 3$	open, contractible	yes	uniformly positive scalar curvature	Chang–Weinberger–Yu
			nonnegative scalar curvature, bounded geom.	Chodosh–Lai–Xu
$n = 4$	open, contractible, tame, Mazur	yes	uniformly positive scalar curvature	Chodosh–Máximo–Mukherjee

Definition

A Mazur manifold is a compact, contractible smooth 4-manifold with boundary admitting a handle body decomposition with one 0-handle, one 1-handle, and one 2-handle.

Question I in 5D

Theorem A (S.)

Let M^5 be the interior of a compact, contractible 5-manifold with boundary X, such that $\pi_3(X, \partial X) = 0$. If M supports a complete metric of uniformly positive scalar curvature, then M is diffeomorphic to $\mathbb{R}^5$.

A word on completeness of the metric

- For every manifold M that is the interior of a compact manifold with boundary, we know, by Gromov's h-principle, that M supports a (*possibly incomplete*) Riemannian metric with positive sectional curvature.
- Therefore, without the completeness assumption one cannot hope for any restrictions on the topology.

Contractible Manifolds I

Proposition

Let X be a compact contractible $(n+1)$-manifold with boundary. Then ∂X is a homology n-sphere, i.e., $H_*(\partial X) = H_*(\mathbb{S}^n)$.

Theorem (Kervaire)

Every 4-homology sphere bounds a contractible manifold. If Σ is a smooth oriented homology n-sphere, $n \geq 5$, then there exists a unique smooth homotopy sphere S^n_Σ such that $\Sigma^n \# S^n_\Sigma$ bounds a contractible smooth manifold.

Remark

When $n = 5$, by the resolution Poincaré Conjecture, a smooth oriented homology 5-sphere bounds a contractible manifold.

Contractible Manifolds II

Theorem (Kervaire)

Let π be a finitely presented superperfect group (i.e, $H_1(\pi) = 0$ and $H_2(\pi) = 0$, where $H_i(\pi)$ denoted the ith homology group of π with coefficients in the trivial $\mathbb{Z}\pi$-module $\mathbb{Z}$). Then for $n \geq 5$ there exists a homology n-sphere Σ^n such that $\pi_1(\Sigma) = \pi$.

Theorem

Let π be a finitely presented perfect group (i.e, $H_1(\pi) = 0$) further assume that the presentation has an equal number of generators and relators. Then there exists a homology 4-sphere Σ^4 with $\pi_1(\Sigma^4) = \pi$.

Remark

Consider the binary icosahedral group $2I = \langle s, t | (st)^2 = s^3 = t^5 \rangle$. For any $n \geq 3$, there exists a homology n-sphere Σ^n such that $\pi_1(\Sigma) = 2I$.

Returning to Question I in 5D

Theorem A (S.)

Let M^5 be the interior of a compact, contractible 5-manifold with boundary X, such that $\pi_3(X, \partial X) = 0$. If M supports a complete metric of uniformly positive scalar curvature, then M is diffeomorphic to $\mathbb{R}^5$.

By combining the combining the works of Ratcliffe–Tschantz, Kervaire, and Anderson we have the following:

Remark

There exists infinitely many aspherical smooth homology 4-spheres. Thus, they bound a compact, contractible 5-manifold X. Moreover, $\pi_3(X, \partial X) = 0$.

Conditions for Compact manifolds with boundary to be Diffeomorphic to $\mathbb{D}^{n+1}$

Dimension (of boundary)	**Topology**	**Curvature Condition**	**Boundary Condition**	**References**
$n = 1$	Compact with boundary	positive (scalar) curvature	positive geodesic curvature	Gauss–Bonnet
$n = 2$	Compact with boundary, contractible	positive scalar curvature	mean convex boundary	Carlotto–Li

Compact manifolds with boundary in dimensions ≥ 4

By combining the works of Lawson–Michelsohn, Kervaire, and Bär–Hanke we have the following:

Proposition

*Let X^{n+1}, $n \geq 2$, be a compact, contractible $(n+1)$-manifold with boundary. If $n = 3$, additionally assume that X is a Mazur manifold. Then X supports a Riemannian metric with **positive scalar curvature and mean convex boundary**.*

Proposition

*Let X^{n+1}, $n \geq 2$, be a compact, contractible $(n+1)$-manifold with boundary. If $n = 3$, additionally assume that X is a Mazur manifold. Then X supports a Riemannian metric with **positive scalar curvature and convex boundary**.*

What is known for stronger curvature conditions?

Dimension (of boundary)	**Topology**	**Curvature Condition**	**Boundary Condition**	**References**
$n \geq 1$	Compact with boundary	positive sectional curvature	convex boundary	Soul Theorem of Gromoll–Meyer, Cheeger–Gromoll
$n = 2$	Compact with boundary	positive Ricci curvature	mean convex boundary	Fraser–Li

Other Curvatures

Are there curvature conditions which are **stronger** than the combination of positive scalar curvature on the interior and mean convexity on the boundary, yet **weaker** than positive sectional curvature on the interior and convexity on the boundary, that can distinguish the disk?

Other Curvatures II

A Riemannian manifold (M^n, g), $n \geq 4$, has positive isotropic curvature (PIC) if $R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0$ for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$.

- PIC implies positive scalar curvature.
- We observe that for a Riemannian manifold (M^n, g) with $n \geq 4$, the condition of PIC is *incomparable* with the condition of positive Ricci curvature; neither implies the other, in general.
    - We recall $(\mathbb{S}^1 \times \mathbb{S}^n, g_{\mathbb{S}^1} \oplus g^n e_{rd})$ has PIC but $\mathbb{S}^1 \times \mathbb{S}^n$ does not admit a Riemannian metric with positive Ricci curvature, and $(\mathbb{S}^p \times \mathbb{S}^q, g^p_{rd} \oplus g^q_{rd})$, $p, q \geq 2$, has positive Ricci curvature but $\mathbb{S}^p \times \mathbb{S}^q$ does not admit a Riemannian metric with PIC.

An Answer to Question II

Theorem B (S.)

Let X^{n+1} be a compact, contractible $(n+1)$-manifold with boundary such that one of the following two conditions holds.

(i) $n = 4$ or $n \geq 12$ and X supports a Riemannian metric g with PIC and the boundary is 2-convex.

(ii) $n \in \{3, 4\}$ and X supports a Riemannian metric g such that $ng \leq \mathrm{Ric} \leq \frac{1}{2}n(n+1)g$ and the boundary is convex; furthermore, if $n = 4$, assume $\pi_3(X, \partial X) = 0$.

Then X is homeomorphic to the $(n+1)$-disk.

The homeomorphism can be promoted to a diffeomorphism in any of the following cases: when $n = 3$ and X is a Mazur manifold, when $n = 4$ and X supports a Riemannian metric g with PIC and the boundary is 2-convex, and when $n \geq 12$.

Algebraic Topology

Lemma

Let X^{n+1} *be a compact, contractible* $(n+1)$*-manifold with boundary, then* ∂X *is an integral homology sphere, namely,* $H_*(\partial X;\mathbb{Z}) = H_*(\mathbb{S}^n;\mathbb{Z})$.

Sketch of Proof: Use the long exact sequence for homology of pairs.

Lemma

Let M^{2n} *be an integral homology* $2n$*-sphere. Then a finite cover of* M *cannot be homotopy equivalent to a connected sum of finitely many copies of* $\mathbb{S}^{2n-1} \times \mathbb{S}^1$.

Sketch of Proof: Use the Euler Characteristic.

Theorem (Sjerve)

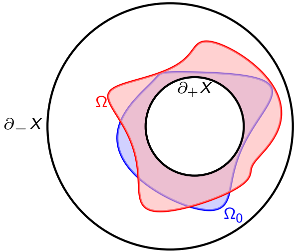
If M^n, $n \geq 3$, *is an integral homology* n*-sphere which is covered by* $\mathbb{S}^n$, *then either* $\pi_1(M) = 0$ *or* $n = 3$ *and* $\pi_1(M) = 2I$ *(i.e., the Poincaré homology 3-sphere).*

μ-Bubbles

Let (X^{n+1}, g), $2 \leq n \leq 6$, be a Riemannian $(n+1)$-manifold with boundary. Assume that $\partial X \neq \emptyset$ and that ∂X has at least two boundary components. Now let $\partial X = \partial_- X \sqcup \partial_+ X$ where both $\partial_+ X$ and $\partial_- X$ are nonempty and both $\partial_+ X$ and $\partial_- X$ are unions of connected components of the boundary. Now, fix a function $h \in C^\infty(\mathrm{int}(X))$ such that $h \to +\infty$ on $\partial_+ X$ and $h \to -\infty$ on $\partial_- X$. Now, choose an open Caccioppoli set Ω_0 with smooth boundary $\partial \Omega_0 \subset \mathrm{int}(X)$ and $\partial_+ X \subset \Omega_0$. Consider the following functional:

$$\mathcal{A}(\Omega) = \int_{\partial^* \Omega} d\mathcal{H}^n - \int_X \left(\chi_\Omega - \chi_{\Omega_0}\right) h \, d\mathcal{H}^{n+1} \tag{1}$$

for all $\Omega \in \mathcal{C} := \{\text{Caccioppoli sets } \Omega \subset M \text{ with } \Omega \triangle \Omega_0 \Subset \mathrm{int}(X)\}$.



μ-bubbles (Cont.)

For $2 \leq n \leq 6$, there exists a smooth $(n+1)$-manifold which minimizes $\mathcal{A}$ on Ω. We call such a minimizer a *μ-bubble*.
Using μ-bubble one can prove the following Separation Theorem of Gromov.

Theorem (Gromov)

Fix a constant $\kappa > 0$. Let (X^{n+1}, g), $2 \leq n \leq 6$, be a Riemannian $(n+1)$-manifold with boundary. Assume that $\partial X \neq \emptyset$ and that ∂X has at least two boundary components. Let $\partial X = \partial_- X \sqcup \partial_+ X$ where $\partial_\pm X \neq \emptyset$ are unions of connected components of the boundary. Assume that the scalar curvature satisfies $\mathrm{R}_X \geq \kappa > 0$. Then there is a constant $C(\kappa) = \max\left\{3\pi, \frac{5\pi}{2\kappa}\right\}$ such that if $d(\partial_- X, \partial X_+) > C$, there exists a smooth embedded closed 2-sided hypersurface $N \subset \mathrm{int}(X)$ that separates $\partial_- X$ from $\partial_+ X$ and supports a Riemannian metric with positive scalar curvature.

Key Proposition

Proposition

Let X^5 be a compact, contractible 5-manifold with boundary such that the interior of X supports a complete Riemannian metric with uniformly positive scalar curvature. Further assume $\pi_3(X, \partial X) = 0$. Then the boundary ∂X has a finite cover that is homotopy equivalent to $\mathbb{S}^4$ or a connected sum of finitely many copies of $\mathbb{S}^3 \times \mathbb{S}^1$.

Sketch of Proof.

i) As X is contractible and $\pi_3(X, \partial X) = 0$, we conclude that $\pi_2(X) = 0$.

ii) By Gromov's Separation Theorem there exists a hypersurface N in a neighborhood of infinity ($\cong \partial X \times [-1, 1]$).

iii) Let $\rho : N \to \partial X$ be the restriction of $\mathrm{proj} : \partial X \times [-1, 1] \to \partial X \times \{1\}$. Note ρ is degree non-zero.

iv) Thus, ∂X has a finite cover that is homotopy equivalent to $\mathbb{S}^4$ or a connected sum of finitely many copies of $\mathbb{S}^3 \times \mathbb{S}^1$ by Chodosh–Li–Liokumovich.

Proof of Theorem A

Theorem A (S.)

Let M^5 be the interior of a compact, contractible 5-manifold with boundary X, such that $\pi_3(X, \partial X) = 0$. If M supports a complete metric of uniformly positive scalar curvature, then M is diffeomorphic to $\mathbb{R}^5$.

Sketch of Proof.

i) By the previous proposition, ∂X has a finite cover that is homotopy equivalent to $\mathbb{S}^4$ or a connected sum of finitely many copies of $\mathbb{S}^3 \times \mathbb{S}^1$.

ii) By Algebraic Topology, we conlcude that ∂X is a homology 4-sphere and so cannot be covered a connected sum of finitely many copies of $\mathbb{S}^3 \times \mathbb{S}^1$.

iii) By Algebraic Topology, we have that ∂X is a simply connected integral homology sphere.

iv) Now by the Hurewicz Theorem, ∂X is a homotopy 4-sphere and so by Freedman is homeomorphic to $\mathbb{S}^4$.

v) By the resolution of the Poincaré conjecture and a result of Stalling we have M is diffeomorphic to $\mathbb{R}^5$.

Thank you!

Proof of One Case of Theorem B

Theorem B (S.)

Let X^5 be a compact, contractible 5-manifold with boundary such that X supports a Riemannian metric g with PIC and the boundary is 2-convex. Then X is diffeomorphic to the 5-disk.

Sketch of Proof.

i) ∂X admits a metric with PIC by a result of Chow and a computation.

ii) Thus, ∂X is diffeomorphic to $\mathbb{S}^4 \# (\#_{j=1}^{J} \mathbb{S}^4/\Gamma_j) \# (\#_{k=1}^{K} (\mathbb{S}^3 \times \mathbb{R})/G_k)$ by Chen–Tang–Zhu.

iii) $H_1(\partial X) = \Gamma_1 * \cdots * \Gamma_J * G_1 * \cdots * G_K$.

iv) For all j, $\mathbb{S}^4/\Gamma_j \cong \mathbb{RP}^4$ and so $\Gamma_j = \mathbb{Z}_2$. Also, G_k are virtually infinitely cyclic.

v) As $H_1(\partial X) = 0$, thus $J = 0$. We also note that virtually infinitely cyclic groups have nontrivial abelianizations so $K = 0$.

vi) Thus ∂X is diffeomorphic to $\mathbb{S}^4$.

vii) By the resolution of the Poincaré conjecture X is diffeomorphic to $\mathbb{D}^5$.