Asymptotics of shortest filling closed geodesics

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Contents

Outline for section 1

Basic definitions

- A genus g closed hyperbolic surface is a genus g closed surface with a metric of constant curvature -1.
- Moduli space of hyperbolic surfaces $\mathcal{M}_g$: the space of all genus g closed hyperbolic surfaces up to isometry.
- Teichmüller space $\mathcal{T}_g$: universal cover of $\mathcal{M}_g$, consisting of all hyperbolic structures on genus-g surface up to isotopy.
- $\mathcal{T}_g$ is homeomorphic to $\mathbb{R}^{6g-6}$ and $\mathcal{M}_g \cong \mathcal{T}_g/\Gamma_g$.

Filling (multi-)geodesics on hyperbolic surfaces

A set of geodesics on a closed hyperbolic surface is filling if it cuts the surface into polygons.

- Thurston initiated the study on filling closed multi-geodesics in several work, concerning Nielsen realization theorem (Kerckhoff etc.), the construction of pseudo-Anosov mappings by the Dehn twists on a filling pair, the construction of Thurston spine.
- Topics about filling closed multi-geodesics: the mapping class group and curve complex (Masur-Minsky etc), systole function (Schmutz etc), asymptotics of long filling geodesics on random hyperbolic surfaces (Wu-Xue, Dozier-Sapir).

The function $\mathcal{L}_{\text{sys}}^{\text{fill}}(X_g)$

For the genus g hyperbolic surface X_g, define $\mathcal{L}_{\text{sys}}^{\text{fill}}(X_g)$ as the minimal length of filling closed multi-geodesics on X_g. The quantity

$$\mathcal{L}_{\text{sys}}^{\text{fill}} : \mathcal{M}_g \to \mathbb{R}^{>0}$$

gives a natural function on $\mathcal{M}_g$. It is proper by the Collar Lemma. Very recently, it is obtained that

Theorem 1 (Gao-Wang-Wang (2025+))

$$\min_{X_g \in \mathcal{M}_g} \mathcal{L}_{\text{sys}}^{\text{fill}}(X_g) = (8g-4)\,\text{arccosh}\left(\sqrt{2}\cos\left(\frac{\pi}{8g-4}\right)\right),$$

that is half of the perimeter of the regular right-angled hyperbolic $(8g-4)$-gon.

Main Theorem 1: Uniform asymptotics of $\mathcal{L}_{\text{sys}}^{\text{fill}}(X_g)$

The function $\mathcal{L}_{\text{sys}}^{\text{fill}}(X_g)$ is unbounded as X_g approaches the boundary of $\mathcal{M}_g$ that is equivalent to saying that the systole of X_g goes to 0. In this work, we study its uniform asymptotic behavior as systole $\to 0$. Our first main result is as follows.

Theorem 2 (Gao-Wang-Wu)

Let X_g be a closed hyperbolic surface of genus g. Then

$$\frac{7}{2}g + \mathrm{R}(X_g) \leq \mathcal{L}_{\text{sys}}^{\text{fill}}(X_g) \leq 300g + 12\mathrm{R}(X_g)$$

where

$$\mathrm{R}(X_g) = \sum_{\textit{closed geodesic } \gamma \subset X_g, \ \ell(\gamma) < 1} \log\left(\frac{1}{\ell(\gamma)}\right).$$

The lower bound of Theorem 2 follows directly from Theorem 1 and the Collar lemma. The difficult part is the upper bound. Two major new ingredients to obtain the upper bound:

- the existence of a shortest embedded filling graph;
- its dual 4-valent graph.

WP random surfaces I

- On $\mathcal{T}_g$ and $\mathcal{M}_g$, there is a Riemann metric, the Weil-Petersson metric. Mirzakhani gave a celebrated formula on how to calculate integrations on $\mathcal{M}_g$.
- Based on this, Mirzakhani initiated the study of Weil-Petersson random hyperbolic surfaces. For $B \subset \mathcal{M}_g$, she defined the probability of B as

$$\mathrm{Prob}_{\mathrm{WP}}^{g}(B) = \frac{\mathrm{Vol}_{\mathrm{WP}}(B)}{\mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_g)}.$$

WP Random surfaces II

This subject has become very active in recent years. Some surprising geometric properties and its connection to other subjects have been established. We list some results on geometric properties on this model:

- Bers constant, for random surfaces (Guth-Parlier-Young 2011);
- Injectivity radius, Cheeger constant etc for random surfaces, thin part volume (Mirzakhani 2013);
- Geodesic length distribution (Mirzakhani-Petri 2019);
- Separating systole, for random surfaces (Nie-Wu-Xue 2023), expectation (Parlier-Wu-Xue 2022);
- Tangle-free hypothesis (Monk-Thomas 2022);
- Asymptotics of long filling geodesics (Wu-Xue 2025, Dozier-Sapir 2023+);
- Non-simple systoles for random surfaces (He-Shen-Wu-Xue 2023+);
- Prime geodesic theorem (Wu-Xue 2025).

For volume calculation and random theory on the moduli space of flat surfaces and their relation with the Weil-Petersson geometry, see, for example,

- Calculating $\mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_g)$ (Mirzakhani-Zograf 2015);
- Calculating the volume of the moduli space of flat surfaces (Aggarwal 2020);
- Saddle connection length distribution in the moduli space of flat surfaces (Masur-Rafi-Randecker 2024+).

Main theorems: WP random surfaces

We now view the quantity $\mathcal{L}_{\text{sys}}^{\text{fill}}(X_g)$ as a positive random variable on $\mathcal{M}_g$. As an application of Theorem 1 and 2, we prove

Theorem 3 (Gao-Wang-Wu)

For any $C > 300$,

$$\lim_{g\to\infty} \text{Prob}_{\text{WP}}^{g}\left(X_g \in \mathcal{M}_g;\ \frac{7}{2}g \le \mathcal{L}_{\text{sys}}^{\text{fill}}(X_g) \le Cg\right) = 1.$$

For the expected value, we prove the following.

Theorem 4 (Gao-Wang-Wu)

For any $p > 0$, there exist two positive constants $C_1 = C_1(p)$ and $C_2 = C_2(p)$ only depending on p such that for g large enough,

$$C_1 < \frac{\mathbb{E}_{\text{WP}}^{g}\left[(\mathcal{L}_{\text{sys}}^{\text{fill}})^p\right]}{g^p} < C_2.$$

If $p = 1$, $C_1 = \frac{7}{2}$ and $C_2 > 300$ is arbitrary.

Brooks-Makover model

For $N \geq 2$, take $2N$ hyperbolic ideal triangles and glue them together into a cusped hyperbolic surface, where the gluing scheme is given by a random oriented trivalent graph. The glue is required to have zero shearing coordinates. Denote the gluing pattern as ω and the set of all the gluing patterns as Ω_N. For $\omega \in \Omega_N$, the cusped hyperbolic surface obtained by gluing ideal triangles following the pattern ω is denoted by $S_O(\omega)$. Its compactification by filling all the cusps is denoted by $S_C(\omega)$. The set Ω_N is finite, so define the probability of its subset $B \subset \Omega_N$ as

$$\mathrm{Prob}_{\mathrm{BM}}^{N}(B) = \frac{|B|}{|\Omega_N|}.$$

Results on Brooks-Makover model:

- Model construction (Brooks-Markover 2004);
- Genus (Gamburd 2006);
- Systole (Petri 2015, Liu-Petri 2023+);
- Diameter (Budzinski-Curien-Petri 2021);
- Cheeger constant (Shen-Wu 2023).

Main theorem: BM random surface

Our result on $\mathcal{L}_{\text{sys}}^{\text{fill}}$ for this random model is as follows.

Theorem 5

For any $\omega \in \Omega_N$*,*

$$\pi N \leq \mathcal{L}_{\text{sys}}^{\text{fill}}(S_O(\omega)) \leq 6N.$$

For the compact case,

$$\lim_{N \to \infty} \text{Prob}_{\text{BM}}^{N} \left(\omega \in \Omega_N;\ \frac{7}{2} N < \mathcal{L}_{\text{sys}}^{\text{fill}}(S_C(\omega)) < 6N \right) = 1.$$

For every cusped surface $S_O(\omega)$, we obtain bounds of $\mathcal{L}_{\text{sys}}^{\text{fill}}$. For the conpact surface $S_C(\omega)$, the bounds are for 'most surfaces'.

Outline for section 2

A brief sketch

- Theorem 3 and 4 follow from Theorem 2 and Mirzakhani integration formula.
- To show the upper bound in Theorem 2, namely $\mathcal{L}_{\mathrm{sys}}^{\mathrm{fill}}(X_g) \leq 300g + 12\mathrm{R}(X_g)$ is to show for any $X_g \in \mathcal{M}_g$, there is a filling multi-geodesic on X_g, whose length is bounded above by $300g + 12\mathrm{R}(X_g)$.
- There is a filling graph G with length $\leq 150g + 6\mathrm{R}(X_g)$, see *e.g.* Buser. We try to construct a filling multi-geodesic with similar length from G.
- Any filling graph $G \subset X_g$ induces a filling 4-valent graph $\mathcal{D}(G) \subset X_g$. We call $\mathcal{D}(G)$ the dual 4-valent graph of G. It can be treated as a multi-curve on X_g. But in general, $\mathcal{D}(G)$ is not homotopic to a filling geodesic.
- Let $\mathcal{G}$ be a family of filling geodesic graphs with a bounded number of vertices and edges. **If $G_0 \in \mathcal{G}$ is the graph with the minimal length in $\mathcal{G}$, then $\mathcal{D}(G_0)$ is isotopic to a filling set of closed geodesics on X_g.** Moreover,

$$\ell(G_0) \leq \ell(\mathcal{D}(G_0)) \leq 2\ell(G_0).$$

- **The minimal-length graph in $\mathcal{G}$ exists. Moreover, it has good geometric and combinatorial properties.**

Filling graph with minimal length

Proposition 6

There exists an embedded graph $G_0 \in \mathcal{G}$ such that

$$\ell(G_0) = \min_{G \in \mathcal{G}} \ell(G).$$

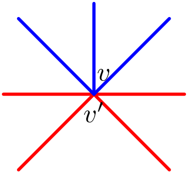
Proposition 7

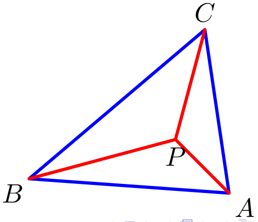
For $G_0 \in \mathcal{G}$ with $\ell(G_0) = \min_{G \in \mathcal{G}} \ell(G)$, then

1. *G_0 is trivalent.*
2. *At each vertex, the angle between any two edges equals $\frac{2}{3}\pi$.*
3. *$X \setminus G_0$ consists of exactly one disk.*

Outline of proofs

- We prove Proposition 6 by showing the subsequence-convergence of length-minimizing sequences in $\mathcal{G}$. The difficulty is that embedded graphs may converge to a non-embedded graph.
- To overcome this, we show that for any non-embedded graph, one can find a modification, shortening the length of the graph and eliminating some intersections. Moreover, we show one can get an embedded graph after finitely many times of the modifications.
- One of the modifications uses the Fermat point of triangles. The $\frac{2}{3}\pi$ in Proposition 7 comes from the Fermat point. For Fermat point in a hyperbolic triangle, see Tan-Xu.
- Proposition 6 is comparable to Meeks-Simon-Yau's result on minimal surfaces in 3-manifolds. They showed in a 3-manifold, an area-minimizing sequence of embedded surfaces converges to an embedded minimal surface with similar topology to the original sequence of surfaces.





Dual 4-valent graphs

Definition 8

Let $G \in \mathcal{G}$ be a graph embedded in X, with geodesic edges, satisfying that $X \backslash G$ are convex polygons. Join the mid-points of neighboring edges of G by geodesics, then we get a 4-valent graph $\mathcal{D}(G)$. We define $\mathcal{D}(G)$ as the *dual 4-valent graph* of G.

In general, $\mathcal{D}(G)$ may contain bigons, hence may be not filling.

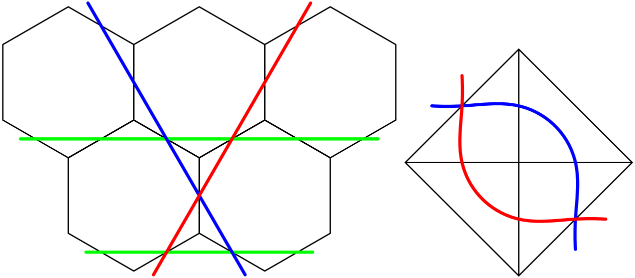


Figure 1: G and $\mathcal{D}(G)$

Dual 4-valent graph of the minimal-length graph

Definition 9 (Minimal position)

For a genus g closed surface Σ, let $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ be a set of incontractible closed curves. Say Γ is *in minimal position* if for any $i, j = 1, 2, ..., n$, the number of intersection points of γ_i and γ_j is minimal in the isotopy classes of γ_i and γ_j.

In other words, if Γ is in minimal position, then no self-intersection point of Γ can be eliminated by homotopy.

Proposition 10

If G_0 is the graph that realizes the minimal length in $\mathcal{G}$, then

1. *each closed curve in its dual 4-valent graph $\mathcal{D}(G_0)$ is homotopically non-trivial.*
2. *The curves in $\mathcal{D}(G_0)$ are in minimal position.*
3. *The dual 4-valent graph $\mathcal{D}(G_0)$ is isotopic to a filling set of closed geodesics in X.*

Minimal position

A creterion of minimal position

Theorem 11 (Hass-Scott, also Gaster)

A curve set $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ is in minimal position if and only if no γ_i bounds an immersed monogon and no pair γ_i, γ_j bounds an immersed bigon.

For contractible closed curves we have

Theorem 12 (Hass-Scott)

A contractible closed curve in a disk either bound a disk or bound an embedded monogon or an embedded bigon.

For curves in minimal position, we have

Theorem 13 (Hass-Scott, also Levitt-Vogtmann)

For $X \in \mathcal{M}_g$ and a set $\Gamma \subset X$ of closed curves in minimal position, there is an isotopy of X, mapping Γ to a set of geodesics on X up to finitely many triangle moves.

We prove Proposition 10 by showing that in the universal cover of X_g, the preimage of $\mathcal{D}(G_0)$ does not bound a disk or a monogon or a bigon by Gauss-Bonnet theorem.

A monogon

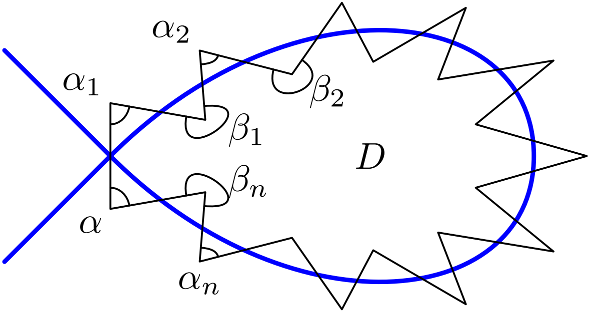


Figure 2: The monogon and D

A generalization to Proposition 10

Proposition 14

Let $X \in \mathcal{M}_g$ and $\mathcal{T}$ be a geodesic triangulation of X. If the degree of every vertex of $\mathcal{T}$ is at least 6, then

1. *each closed curve in its dual 4-valent graph $\mathcal{D}(\mathcal{T})$ is homotopically non-trivial.*
2. *The curves in $\mathcal{D}(\mathcal{T})$ are in minimal position.*
3. *The dual 4-valent graph $\mathcal{D}(\mathcal{T})$ is isotopic to a filling set of closed geodesics in X.*

- It is an essential tool to prove Theorem 5 (compact case), the theorem in the Brook-Makover model.
- The degree of triangulation $\mathcal{T}$ may be explained as the curvature of the surface and $\mathcal{D}(\mathcal{T})$ may be explained as geodesics in this curvature. An naive example is given by the triangulation of $\mathbb{R}^2$ by regular triangles whose degree is 6 at every vertex and whose dual 4-valent graph consists of geodesics in $\mathbb{R}^2$. Another example is about $\mathbb{H}^2$. In $\mathbb{H}^2$, regular triangulation always has degree > 6, and its dual 4-valent graph also consists of geodesics in $\mathbb{H}^2$.

Thank you!

Thank you!