A Little Bit of Topology Disguised as Analysis

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The Exterior Derivative

Recall

(For simplicity) let M be a smooth closed manifold: it is compact and has no boundary. Denote by $\Omega^p(M)$ the space of smooth $p-$forms on M. Recall that there is a unique linear map $d : \Omega^p(M) \to \Omega^{p+1}(M)$, called the exterior derivative, which satisfies the following three properties.

1. Recall that $\Omega^0(M)$ is the collection smooth functions on M. Then, for $f \in \Omega^0(M)$ the element $df \in \Omega^1(M)$ is the usual differential of f: for all $w \in T_pM$ we have $df(w) = w(f)$.
2. For $\eta \in \Omega^p(M)$ and $\sigma \in \Omega^q(M)$ we have $d(\eta \wedge \sigma) = (d\eta) \wedge \sigma + (-1)^p \eta \wedge (d\sigma)$.
3. The map $d \circ d : \Omega^p(M) \to \Omega^{p+2}(M)$ is the zero map: we have $d \circ d = d^2 = 0$.

The de Rham Cohomology

Recall

Let M be a smooth closed manifold, and fix p. Set
$B = \{\omega \in \Omega^p(M) : d\omega = 0\}$ and
$Z = \{\omega \in \Omega^p(M) : \exists \sigma \in \Omega^{p-1}(M), d\sigma = \omega\}$. We define

$$H^p_{dR}(M) = \frac{B}{Z}. \tag{1}$$

Line Integrals

Recall

Let M be a closed smooth manifold, let $\omega \in \Omega^1(M)$, and let $\gamma : [0,1] \to M$ be a smooth curve. Recall that the line integral of ω along γ is defined to be

$$\int_\gamma \omega = \int_0^1 \gamma^* \omega. \tag{2}$$

Smooth Curves are Dense

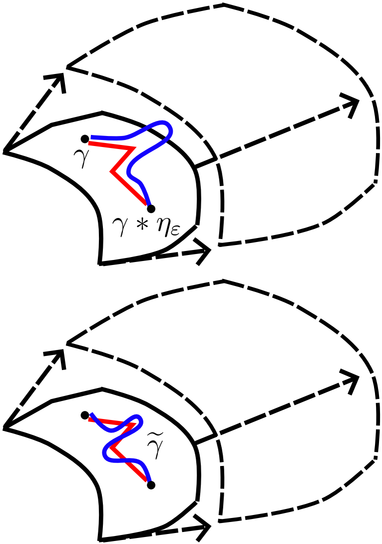
Lemma

Let M be a smoot closed manifold, and let $\gamma : [0,1] \to M$ be a continuous curve. Then, there is a smooth curve $\widetilde{\gamma} : [0,1] \to M$ whose endpoints agree with γ. Furthermore, this curve is homotopic to γ with respect to its endpoints.

Proof.

An application of the Whitney Approximation Theorem. □

Sketch



On Smooth Manifolds, Smooth Curves are Enough

Observation

The Whitney approximation means that for a smooth closed manifold M, if we wish to understand $\pi_1(M)$ we may restrict our attention to smooth curves, and smooth homotopies.

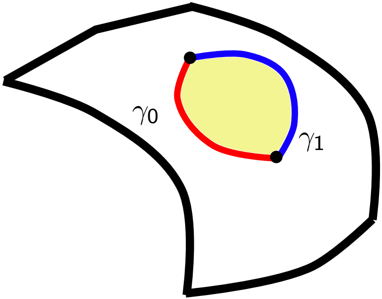
Homotopy Invariance

Lemma

Let M be a closed smooth manifold, let γ_0 and γ_1 be two smooth curves with the same endpoints, and let $F : [0,1] \times [0,1] \to M$ be a smooth homotopy between them which fixes the endpoints. If $\omega \in \Omega^1(M)$ is such that $d\omega = 0$, then

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega. \tag{3}$$

Sketch



$$0 = \int_{I\times I} F^* d\omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega$$

Integral Cohomology Lattice

Definition

Let M be a smooth closed manifold, let $H^1_{dR}(M)$ be the first de Rham cohomology, and let $\pi_1(M)$ be the fundamental group where we are only considering smooth loops and smooth homotopies. Denote by $H^1(M;\mathbb{Z})_{\mathbb{R}}$ the following subset:

$$H^1(M;\mathbb{Z})_{\mathbb{R}} = \left\{ [\omega] \in H^1_{dR}(M) : \int_\gamma \omega \in \mathbb{Z}, \forall \gamma \in \pi_1(M) \right\}. \tag{4}$$

We call $H^1(M;\mathbb{Z})_{\mathbb{R}}$ the integral cohomology lattice.

Maps to $\mathbb{S}^1$

Lemma

Let M be a smooth closed manifold, and let $H^1(M;\mathbb{Z})_{\mathbb{R}}$ be the integral cohomology lattice. For every $\omega \in \Omega^1(M)$ such that $[\omega] \in H^1(M;\mathbb{Z})_{\mathbb{R}}$ we get a smooth map $f_\omega : M \to \mathbb{S}^1$.

Proof.

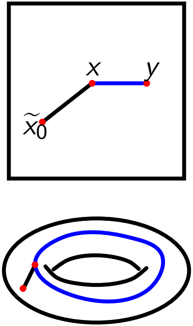
Let $\widetilde{M}$ be the universal cover, let $\pi : \widetilde{M} \to M$ be the covering map, and fix $\widetilde{x}_0$ in $\widetilde{M}$. Then, the map $\widetilde{f}_\omega(x) = \int_{\gamma_x} \pi^*\omega$ is a well defined map from $\widetilde{M}$ to $\mathbb{R}$.

Proof.

Suppose that $\pi(x) = \pi(y)$, and let γ_{xy} be a curve from x to y; the curve $\gamma_y = \gamma_{xy} \cdot \gamma_x$ goes from $\widetilde{x}_0$ to y. We have

$$\widetilde{f}_\omega(y) - \widetilde{f}_\omega(x) = \int_{\gamma_{xy}} \pi^*\omega = \int_{\pi(\gamma_{xy})} \omega \in \mathbb{Z}.$$

□



The Best Representative

Observation

Since representatives ω of elements of $H^1(M;\mathbb{Z})_{\mathbb{R}}$ generate maps $f_\omega : M \to \mathbb{S}^1$, we may expect that well behaved representatives give well behaved maps.

Raising and Lowering

Definition

Let (M, g) be a smooth Riemannian manifold, and let $p \in M$. Then given $v \in T_pM$ we let $v^\flat$ be the unique covector in T_p^*M such that for all $w \in T_p(M)$ we have

$$v^\flat(w) = g(w, v). \tag{5}$$

Similarly, given $\sigma \in T_p^*M$ we let $\sigma^\sharp$ be the unique vector in T_pM such that for all vectors $w \in T_pM$ we have

$$g(\sigma^\sharp, w) = \sigma(w). \tag{6}$$

Raising and Lowering Tensors

Lemma

Let (M, g) be a smooth Riemannian manifold, and let $p \in M$. Then g gives rise to maps we get maps $\flat : \bigotimes_{i=1}^{r} T_pM \to \otimes T_p^*M$ and $\sharp : \bigotimes_{i=1}^{r} T_p^*M \to \otimes T_pM$

Proof.

1. For $\nu = v_1 \otimes \cdots \otimes v_r$ set $\nu^\flat = v_1^\flat \otimes \cdots \otimes v_r^\flat$
2. Extend $\flat$ by linearity.

$\square$

Contraction

Definition

Let M be a smooth manifold, let $p \in M$, and let $k \in \mathbb{N}$. Suppose we have $\sigma = \sigma_1 \otimes \cdots \sigma_k \in \bigotimes_{i=1}^k T_P^* M$ and $\nu = \nu_1 \otimes \cdots \otimes \nu_k \in \bigotimes_{i=1}^k T_p M$ Then, we define $\langle \sigma, \nu \rangle = \sigma_1(\nu_1) \cdots \sigma_k(\nu_k)$. This pairing is bi-linear in σ and ν, and so extends to a bilinear map

$$\langle \cdot, \cdot \rangle : \bigotimes_{i=1}^{k} T_p^* M \times \bigotimes_{i=1}^{k} T_p M \to \mathbb{R}. \tag{7}$$

Innerproduct on Tensors

Definition

Let (M, g) be a smooth Riemannian manifold, let $p \in M$, and let $k \in \mathbb{N}$. Given $v, w \in \bigotimes_{i=1}^{k} T_pM$ we define

$$g_p(v, w) = \langle v^\flat, w \rangle. \tag{8}$$

Definition

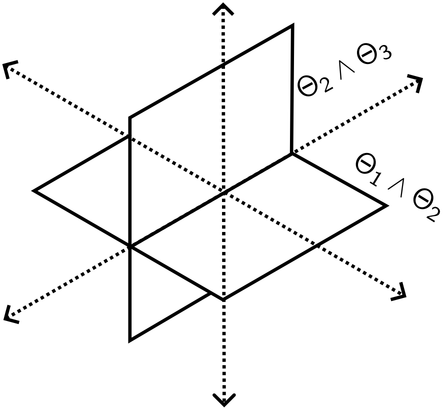
Let (M, g) be a smooth Riemannian manifold, let $p \in M$, and let $k \in \mathbb{N}$. Given $\sigma, \eta \in \bigotimes_{i=1}^{k} T_p^*M$ we define

$$g_p(\sigma, \eta) = \langle \sigma, \eta^\sharp \rangle. \tag{9}$$

Innerproduct on Forms

Definition

Let (M, g) be a smooth Riemannian manifold, let $p \in M$, and let $k \in \mathbb{N}$. The innerproduct g_p on $\bigotimes_{i=1}^{k} T_pM$ and $\bigotimes_{i=1}^{k} T_p^*M$ descends to an innerproduct g_p on $\bigwedge_{i=1}^{k} T_pM$ and $\bigwedge_{i=1}^{k} T_p^*M$.


$\Theta_2 \wedge \Theta_3$
$\Theta_1 \wedge \Theta_2$

The Orthonormal Basis

Definition

Let m and k be natural numbers. Let us denote by $I(k,m)$ the set

$$I(k,m) = \Big\{ (n_i)_{i=1}^{k} : n_i \in \{1,\dots,m\} \Big\}. \tag{10}$$

We call elements of $I(k,m)$ multi-indices.

Definition

Let m and k be natural numbers, and suppose that $k \leq m$. We denote by $J(k,m)$ the set

$$J(k,m) = \Big\{ (n_i)_{1}^{k} : n_i \in \{1,\dots,m\} \text{ and } n_i < n_j \text{ if } i < j \Big\}. \tag{11}$$

We call elements of $J(k,m)$ increasing multi-indices.

The Orthonormal Basis

Lemma

Let (M, g) be a smooth m-dimensional Riemannian manifold, let $p \in M$, and let $k \in \mathbb{N}$. Let $\{E_i\}_1^m$ be an orthonormal basis for T_pM, and let $\{\Theta_i\}_1^m$ be the dual basis for T_p^*M: we have $\Theta_i(E_j) = \delta_{ij}$. For $I \in I(k, m)$ let E_I denote $E_{I_1} \otimes \cdots \otimes E_{I_k}$. Then, the collection $\{E_I\}_{I \in I(k,m)}$ is an orthonormal basis for $\bigotimes_{i=1}^k T_pM$. If $k \leq m$, then $\{\Theta_J\}_{J \in J(k,m)}$ is an orthonormal basis for $\bigwedge_{i=1}^k T_p^*M$.

Innerproduct on Forms

Definition

Let (M, g) be a smooth closed oriented Riemannian manifold. Then, given $\omega, \sigma \in \Omega^k(M)$ we define their innerproduct to be

$$g(\omega, \sigma) = \int_M g_p(\omega_p, \nu_p)\, dV_g(p) \tag{12}$$

Harmonic Representative

Definition

Let (M, g) be a smooth closed oriented Riemannian manifold, and let w be an element of $H^k_{dR}(M)$. Suppose that $\omega \in \Omega^k$ is such that $[\omega] = w$ and

$$\int_M g_p\left(\omega_p, \omega_p\right) dV_g(p) = \min\left\{g(\nu, \nu) : [\nu] = w\right\}. \tag{13}$$

Then, we call ω an harmonic representative of w.

Harmonic Representative

Calculation

Let (M, g) be a smooth closed oriented Riemannian manifold, and let ω be an harmonic representative.

1. For any $\eta \in \Omega^{k-1}$ consider the path $t \mapsto \omega + td\eta$.
2. We have $0 = \frac{d}{dt}\big|_0 \int_M g(\omega + td\eta, \omega + td\eta) dV_g = 2 \int_M g(\omega, d\eta) dV_g$.
3. Taking the adjoint, we have $0 = \int_M g(d^*\omega, \eta) dV_g$ for all η: we must have $d^*\omega = 0$.

Dual to the Exterior Derivative

Definition

Let (M, g) be a smooth closed oriented Riemannian manifold. Then, we define $\delta : \Omega^{k+1}(M) \to \Omega^k(M)$ to be the adjoint of d: given $\omega \in \Omega^{k+1}(M)$ the form $\delta\omega$ is the unique element of $\Omega^k(M)$ such that for all $\eta \in \Omega^k(M)$ we have

$$g(\delta\omega, \eta) = g(\omega, d\eta). \tag{14}$$