Rigidity and Stability of Three-tori

Edward Bryden

University of Antwerp

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1. **Rigidity**

2. **How Different is the Stability Case?**

3. **Producing a Good Map**
   - Successive Minima and Hebda's Lemma
   - Estimates in Terms of the Stable Systoles

Rigidity

Theorem

Let g be a smooth Riemannian metric on $\mathbb{T}^3$. Denote by S the scalar curvature of g. If $S_g \geq 0$, then the metric g is flat.

Proof.

- We have $\mathbb{Z}^3 = H^1(\mathbb{T}^3;\mathbb{Z}) \simeq H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}}$. Let $\{w^i\}_{i=1}^3$ be a basis of $H^1(M;\mathbb{Z})_{\mathbb{R}}$.
- For each i, find an harmonic one-form $\omega^i \in w^i$, and let $u^i : \mathbb{T}^3 \to \mathbb{S}$ be the map generated by ω^i. Let $\mathbb{U} : \mathbb{T}^3 \to \mathbb{T}^3$ be (u^1, u^2, u^3).
- Observe that $\mathbb{U}^*\left(d\theta^1 \wedge d\theta^2 \wedge d\theta^3\right) = du^1 \wedge du^2 \wedge du^3$. Since $du^i = \omega^i \in w^i$, and the w^i form a basis, it follows that $du^1 \wedge du^2 \wedge du^3$ represent a basis element of $H^3(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}}$: so $\int_{\mathbb{T}^3} du^1 \wedge du^2 \wedge du^3 = \pm 1$.

Proof.

- Force $\mathbb{U}$ to have $\deg(\mathbb{U}) = 1$. In particular, we see that $\mathbb{U}$ is surjective.
- Consider the matrix of functions $g_{ij} = g(du^i, du^j)$. Observe that $dg_{ij}(X) = g(\nabla_X du^i, du^j) + g(du^i, \nabla_X du^j)$.
- Take the norm and integrate to obtain

$$\int_{\mathbb{T}^3} |dg_{ij}| \mathrm{vol}_g \leq \int_{\mathbb{T}^3} |\nabla du^i||du^j| + |\nabla du^j||du^i| \mathrm{vol}_g .$$

Proof.

- From Stern's inequality, we have

$$-\int_{\mathbb{T}^3} S|du^i| \geq \int_{\mathbb{T}^3} \frac{|\nabla du^i|^2}{|du^i|} \mathrm{vol}_g.$$

- If $S \geq 0$, this means that $|\nabla du^i| = 0$, and so g_{ij} is constant for all i, j.
- $\det(g_{ij}) = |du^1 \wedge du^2 \wedge du^3|^2$. Since $\int_{\mathbb{T}^3} du^1 \wedge du^2 \wedge du^3 = 1$, this shows that $\det(g_{ij}) > 0$, and $du^1 \wedge du^2 \wedge du^3$ never vanishes.
- Recall that

$$\deg(\mathbb{U}) = \sum_{x \in \mathbb{U}^{-1}\{y\}} \mathrm{sign}(du^1 \wedge du^2 \wedge du^3)(x)$$

Proof.

- Let $a = (g_{ij})^{-1}$, and consider the metric $g_F = a_{kl} d\theta^k d\theta^l$. Then, we see that $\mathbb{U} : (\mathbb{T}, g) \to \left(\mathbb{T}^3, g_F\right)$ is an isometry:

$$\mathbb{U}^* g_F(\nabla u^i, \nabla u^j) = a_{kl} du^k du^l (\nabla u^i, \nabla u^j) = g_{ij} = g\left(\nabla u^i, \nabla u^j\right).$$

□

To Stability

What Changes

- The term $\int_{\mathbb{T}^3} S|du|\mathrm{vol}_g$ will no longer be zero.
- We should only expect integral control: all we can say initially is that $\int_{\mathbb{T}^3} \frac{|\nabla du|^2}{|du|}\mathrm{vol}_g$ is small.
- Controlling $\int_{\mathbb{T}^3} |S||du|\mathrm{vol}_g$ means finding some control on $|du|$.

Goal

Let g be a metric $\mathbb{T}^3$. We wish to find geometric conditions on g which ensure that there exists a basis w^i of $H^1(M;\mathbb{Z})_{\mathbb{R}}$ with harmonic representatives ω^i whose L^2 norms are well controlled.

Definition

Let g be a Riemannian metric on $\mathbb{T}^3$, let $w \in H^1(M;\mathbb{Z})_{\mathbb{R}}$, and let ω be the harmonic representative of w. Then, we define $\|w\|_g = (\int_{\mathbb{T}^3} |\omega|^2 \mathrm{vol}_g)^{\frac{1}{2}}$. This gives $H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}})$ an innerproduct, also denoted g.

Definition

Let g be a Riemannian metric on $\mathbb{T}^3$, then we define $\lambda_1 = \min\left\{|w|_g : w \in H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}}\right\}$, and we call it the first successive minima of the lattice $H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}}$.

Definition

We define

$$\lambda_2 = \min\{\lambda : \exists \text{ independent } v, w \in H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}}, \max\{|v|, |w|\} \leq \lambda\},$$

and λ_3 is defined similarly.

Lemma

There exists a basis of $H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}}$*, say* $\{w^i\}_{i=1}^3$ *such that* $|w^1|_g = \lambda_1$*,* $|w^2|_g \leq \lambda_2$*, and* $|w^3|_g \leq \frac{3}{2}\lambda_3$*.*

Proof.

An application of a few results that can be found in the book by Cassels. □

Remark

These results apply to arbitrary lattices in $\mathbb{R}^n$ with arbitrary norms. In particular, the definition of the successive minima holds for any lattice, and in general we have n of them.

The Lemma of Hebda

Lemma (Hebda)

Let (M, g) be a smooth closed oriented Riemannian manifold. For $a \in H^{m-1}_{dR}(M)$ we have $\mathrm{PD}(a) \in H_1(M; \mathbb{R})$, and

$$\|\mathrm{PD}(a)\|_1 \leq \mathrm{Vol}_g(M)^{\frac{1}{2}} C(m, 1) \min \left\{ \left(\int_M |\omega|^2 \mathrm{vol}_g \right)^{\frac{1}{2}} : \omega \in a \right\}.$$

Lemma

For all w in $H^1(\mathbb{T}^n;\mathbb{Z})_{\mathbb{R}}$, we have that $\mathrm{PD}(w)H_1(\mathbb{T}^n;\mathbb{R})$ has a representative $c = \sum_i z_i\sigma_i$ where the $z_i \in \mathbb{Z}$.

Definition

We set $H_1(\mathbb{T}^n;\mathbb{Z})_{\mathbb{R}} = \{a \in H_1(\mathbb{T}^n;\mathbb{R}) : \exists c = \sum_i z_i\sigma_i \in a, z_i \in \mathbb{Z}\}$. The one stable-systole, denoted $\mathrm{stabsys}_1(\mathbb{T}^n g)$, is defined to be

$$\mathrm{stabsys}_1(\mathbb{T}^n, g) = \inf\left\{\|a\|_1 : a \in H_1(\mathbb{T}^n;\mathbb{Z})_{\mathbb{R}}\right\}.$$

Lemma

Let g be a Riemannian metric on T^3, then we have that

$$\mathrm{stabsys}_1(\mathbb{T}^3, g) \leq \mathrm{Vol}_g(\mathbb{T}^3)^{\frac{1}{2}} C(3,1) \min \left\{ |w|_g : w \in H^2(\mathbb{T}^3, \mathbb{Z})_{\mathbb{R}} \right\}$$

Proof.

Every element $w \in H^1(\mathbb{T}^3; \mathbb{Z})_{\mathbb{R}})$ has Poincaré Dual $\mathrm{PD}(w) \in H_1(\mathbb{T}^3; \mathbb{Z})_{\mathbb{R}}$. Therefore, $\mathrm{PD}(w)$ is a competitor in the definition of $\mathrm{stabsys}_1(\mathbb{T}^3, g)$. It follows from Hebda's result that we have

$$\mathrm{stabsys}_1(\mathbb{T}^3, g) \leq \mathrm{Vol}_g(\mathbb{T}^3)^{\frac{1}{2}} C(3,1) \min \left\{ |w|_g : w \in H^2(\mathbb{T}^3; \mathbb{Z})_{\mathbb{R}} \right\}.$$

□

Definition

We set $H_m(\mathbb{T}^n;\mathbb{Z})_{\mathbb{R}} = \{a \in H_m(\mathbb{T}^3;\mathbb{R}) : \exists c = \sum_i z_i\sigma_i \in a, z_i \in \mathbb{Z}\}$. Then, we define $\mathrm{stabsys}_m(\mathbb{T}^n, g)$ to be

$$\mathrm{stabsys}_1(\mathbb{T}^3, g) = \inf \{\|a\|_m : a \in H_m(\mathbb{T}^n;\mathbb{Z})_{\mathbb{R}}\}.$$

Lemma

Let g be a Riemannian metric on $\mathbb{T}^3$. Then, we have that

$$\mathrm{stabsys}_2(\mathbb{T}^3, g) \leq \mathrm{Vol}_g(\mathbb{T}^3)^{\frac{1}{2}} C(3,2) \min \left\{ |w|_g : w \in H^1(\mathbb{T}^3; \mathbb{Z})_{\mathbb{R}} \right\}$$

Where we are

Let λ_1 denote the first successive minima of $H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}}$ and let μ_1 denote the first successive minima of $H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}}$. Then, we have that

$$\mathrm{stabsys}_1(\mathbb{T}^3, g) \leq C\mathrm{Vol}_g(\mathbb{T}^3)^{\frac{1}{2}}\mu_1$$

and

$$\mathrm{stabsys}_2(\mathbb{T}^3, g) \leq C\mathrm{Vol}_g(\mathbb{T}^3)^{\frac{1}{2}}\lambda_1.$$

Lemma

Let g be a Riemannian metric on $\mathbb{T}^3$. Then, we have that

$$1 = \det_g \left(H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}} \right) \cdot \det_g \left(H^2(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}} \right) \tag{1}$$

Proof.

Let w^i be a basis of $H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}}$ and let θ_j be the dual elements such that $\theta_j(w^i) = \delta_{ij}$. Since the cohomology of $\mathbb{T}^3$ is free, the θ_j are represented by elements $v^j \in H^2(\mathbb{T}^3)$, and since $\star$ is an isometry, we may associate $H^2_{dR}(\mathbb{T}^3)$ with $H^1_{dR}(\mathbb{T}^3)$. □

Lemma

Let $L \subset \mathbb{R}^n$ be a lattice, let $|\cdot|$ be the usual Euclidean norm on $\mathbb{R}^n$, and let $\lambda_1, \ldots, \lambda_n$ be the successive minima of L. Then, there is a constant $C(n)$ depending only on n such that

$$\det(L) \leq \lambda_1 \cdots \lambda_n \leq C(n) \cdot \det(L)$$

Lemma

Let g be a Riemannian metric on $\mathbb{T}^3$ and suppose that $\min\{\mathrm{stabsys}_1(\mathbb{T}^3, g), \mathrm{stabsys}_2(\mathbb{T}^3, g)\} \geq \sigma > 0$. *Then, there is a constant* $C(\sigma, |\mathbb{T}^3|)$ *such that* $\max_{i=1,2,3} \lambda_i \leq C(\sigma, |\mathbb{T}^3|_g)$, *and so there is a basis of* $H^1(\mathbb{T}^3; \mathbb{Z})_{\mathbb{R}})$, *say* $\{w^i\}_{i=1}^3$, *such that* $|w^i|_g \leq C(\sigma, |\mathbb{T}^3|_g)$.

Proof.

Let $\{\lambda_i\}_{i=1}^3$ be the successive minima of $H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}})$ and let $\{\mu_i\}_{i=1}^3$ be the successive minima of $H^2(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}})$. Then, we have

$$\begin{aligned}
\lambda_1^{-3} \geq \lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1} \geq \frac{1}{C\det(H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}})} &= \frac{\det(H^2(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}})}{C} \\
&\geq \frac{\mu_1\mu_2\mu_3}{C^2} \\
&\geq \frac{\mu_1^3}{C^2} \\
&\geq \frac{\mathrm{stabsys}_1^3}{|\mathbb{T}|_g^{\frac{3}{2}} C^2}
\end{aligned}$$

□