Obtaining L^3 Estimates from Stern's Inequality and Integration By Parts

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1. **Successive Minima Drawing**

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3. **L^3 Bounds**
    - Lifting, and Fundamental Domains
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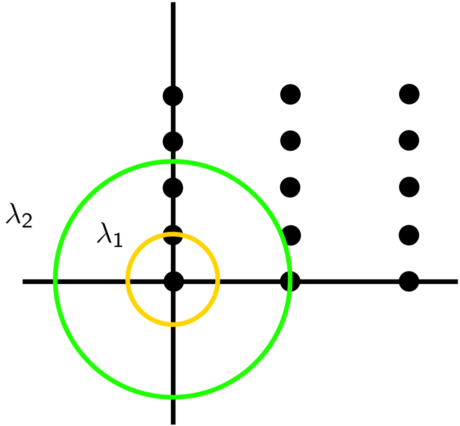
Successive Minima



Lemma

There is a constant $C(V,\sigma)$ *such that for all metrics* g *on* $\mathbb{T}^3$ *with* $\mathrm{vol}_g(\mathbb{T}^3) \geq V$ *and* $\min\left\{\mathrm{stabsys}_1(\mathbb{T}^3,g), \mathrm{stabsys}_2(\mathbb{T}^3,g)\right\} \geq \sigma$, *we can find maps* $u^i : \mathbb{T}^3 \to \mathbb{S}$ *such that* du^i *are harmonic one-forms which form a basis for* $H^1(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}}$ *and which satisfy*

$$\left(\int_{\mathbb{T}^3} |d\omega^i|_g^2 \mathrm{vol}_g\right) \leq C(V,\sigma).$$

Corollary

Let g be a Riemannian metric on $\mathbb{T}^3$ such that $\mathrm{vol}_g(\mathbb{T}^3) \geq V$ and $\min\left\{\mathrm{stabsys}_1(\mathbb{T}^3, g), \mathrm{stabsys}_2(\mathbb{T}^3, g)\right\} \geq \sigma$. Let $S_g^- = \max\{0, -S_g\}$. Then, we have that

$$C(V,\sigma)\|S_g^-\|_{L^2} \geq \int_{\mathbb{T}^3} \frac{|\nabla du^i|^2}{|du^i|}\mathrm{vol}_g.$$

Proof.

Apply Hölder's inequality to Stern's inequality to obtain

$$\|du^i\|_{L^2}\|S_g^-\|_{L^2} \geq \int_{\mathbb{T}^3} |du^i| S_g^- \mathrm{vol}_g \geq \int_{\mathbb{T}^3} \frac{|\nabla du^i|^2}{|du^i|}\mathrm{vol}_g.$$

$\square$

Trial Calculation

Let $g_{ij} = g(du^i, du^j)$, and take the exterior derivative: calculate $dg_{ij}(\cdot) = g(\nabla . du^i, du^j) + g(du^i, \nabla . du^j)$. We get

$$\int_{\mathbb{T}^3} |dg_{ij}| \mathrm{vol}_g \leq \int_{\mathbb{T}^3} \frac{|\nabla du^i|}{|du^i|^{\frac{1}{2}}} |du^j| + \int_{\mathbb{T}^3} \frac{|\nabla du^j|}{|du^j|^{\frac{1}{2}}} |du^i| \mathrm{vol}_g$$

$$\begin{aligned} \int_{\mathbb{T}^3} |dg_{ij}| \mathrm{vol}_g \leq & \left(\int_{\mathbb{T}^3} \frac{|\nabla du^i|^2}{|du^i|} \mathrm{vol}_g \right)^{\frac{1}{2}} \|du^i\|_{L^3}^{\frac{1}{2}} \|du^j\|_{L^3} \\ & + \left(\int_{\mathbb{T}^3} \frac{|\nabla du^j|^2}{|du^j|} \mathrm{vol}_g \right)^{\frac{1}{2}} \|du^j\|^{\frac{1}{2}} \|du^i\|_{L^3}. \end{aligned}$$

Idea

We can rewrite $\int_{\mathbb{T}^3} |du^i|_g^3 \mathrm{vol}_g$ as

$$\int_{\mathbb{T}^3} g(|du^i| du^i, du^i) \mathrm{vol}_g.$$

This formulation lends itself to integration by parts.

Lemma

Let $\mathbb{U} : \mathbb{T}^3 \to \mathbb{T}^3$ *be a map from the three-torus to itself. Then, it lifts to a map* $\hat{\mathbb{U}} : \mathbb{R}^3 \to \mathbb{R}^3$*:*

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "\hat{\mathbb{U}}"] \arrow[d, "\pi"] & \mathbb{R}^3 \arrow[d, "\pi"] \\ \mathbb{T}^3 \arrow[r, "\mathbb{U}"] & \mathbb{T}^3 \end{tikzcd}$$

Proof.

1. Lift $\mathbb{U}$ to a map $\widetilde{\mathbb{U}} : \mathbb{R}^3 \to \mathbb{T}^3$.
2. Then lift $\widetilde{\mathbb{U}}$ to a map $\hat{\mathbb{U}} : \mathbb{R}^3 \to \mathbb{R}^3$.

$\square$

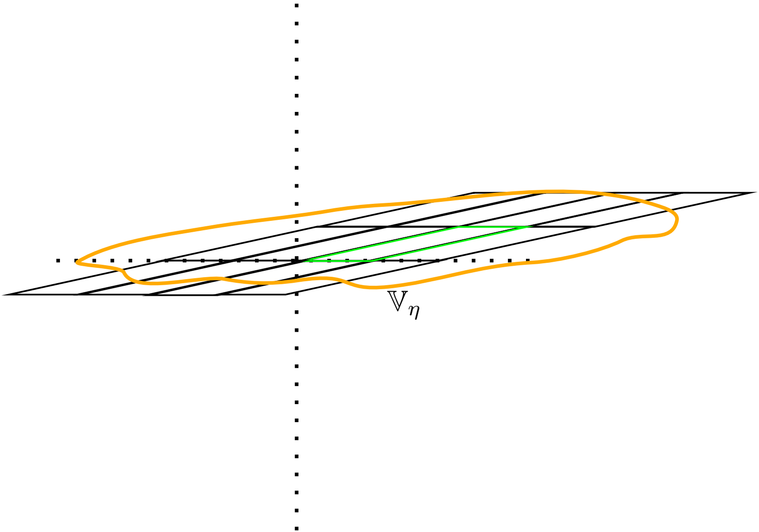
Definition

Let $\pi : \mathbb{R}^3 \to \mathbb{T}^3$ be the covering map. Suppose that $\mathbb{V} \subset \mathbb{R}^3$ is a compact subset of $\mathbb{R}^3$ such that $\pi(\mathbb{V}) = \mathbb{T}^3$, $\pi|_{\mathrm{int}(\mathbb{V})}$ is injective, and $\partial\mathbb{V}$ has measure zero. Then, we call $\mathbb{V}$ a fundamental domain of $\mathbb{T}^3$.

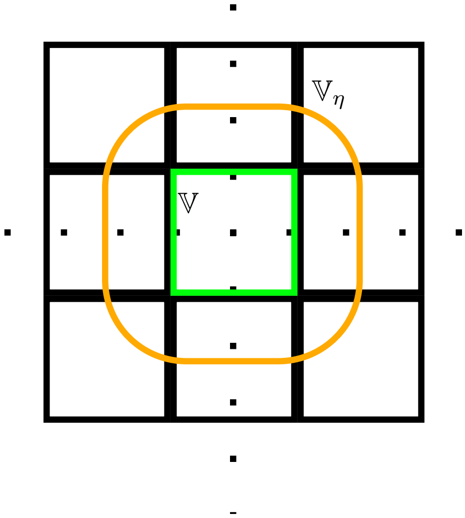
Observation

1. Fundamental domains allow us to move information from $\mathbb{R}^3$ to $\mathbb{T}^3$.
2. Integrating by parts over a fundamental domain might introduce hard to control boundary terms.
3. Use a cutoff, and try to understand how fundamental domains fit next to each other.

Covering



Covering



The Covering Constant

Definition

Let g be a Riemannian metric on $\mathbb{T}^3$ and let $\eta > 0$ be fixed. Then, we define $\kappa(g, \eta)$ as follows

$$\kappa(g,\eta) = \min \left\{ m : \exists \mathbb{V}, \sup_{y \in \mathbb{T}^3} |\pi^{-1}\{y\} \cap \mathbb{V}_\eta| \leq m \right\}.$$

Integration By Parts

Calculation

- Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function cutoff function for $\mathbb{V}_\eta$: $0 \leq f \leq 1$, we have $\operatorname{spt}(f) \subset \mathbb{V}_\eta$, we have $f|_{\mathbb{V}} = 1$, and $|\nabla f|_g \leq \frac{2}{\eta}$.
- Calculate as follows.

$$\int_{\mathbb{R}^3} \pi^* g \left(f |d\hat{u}^i| d\hat{u}^i, d\hat{u}^i \right) \mathrm{vol}_{\pi^* g} = - \int_{\mathbb{R}^3} \hat{u}^i \mathrm{div}_{\pi^* g} \left(f |d\hat{u}^i| d\hat{u}^i \right) \mathrm{vol}_{\pi^* g}$$

- Simplify the divergence term.

$$\mathrm{div}_{\pi^* g} \left(f |d\hat{u}^i| d\hat{u}^i \right) = \pi^* g \left(df, |d\hat{u}^i| d\hat{u}^i \right) + f \pi^* g (d |d\hat{u}^i|, d\hat{u}^i)$$

Integration By Parts

Calculation

- Take norms and estimate to obtain

$$\int_{\mathbb{R}^3} f|d\hat{u}^i|^3_{\pi^* g}\mathrm{vol}_{\pi^* g} \leq \int_{\mathbb{R}^3} |\hat{u}^i| \left(|df||d\hat{u}^i|^2 + f|\nabla d\hat{u}^i||d\hat{u}^i|\right) \mathrm{vol}_{\pi^* g}$$

- Use that f is a cutoff function to obtain

$$\int_{\mathbb{V}} |d\hat{u}^i|^3 \mathrm{vol}_{\pi^* g} \leq \int_{\mathbb{V}_\eta} |\hat{u}^i| \left(\frac{2}{\eta}|d\hat{u}^i|^2 + \frac{|\nabla d\hat{u}^i|}{|d\hat{u}^i|^{\frac{1}{2}}}|d\hat{u}^i|^{\frac{3}{2}}\right) \mathrm{vol}_{\pi^* g}.$$

Sup Bound

Lemma

Let g *be a Riemannian metric on* $\mathbb{T}^3$ *such that* $\mathrm{stabsys}_2(\mathbb{T}^3, g) \geq \sigma$, *and let* $\eta > 0$. *Let* $\mathbb{V}$ *be a fundamental domain which gives* $\kappa(g, \eta)$. *Then, we have that*

$$\sup_{\mathbb{V}_\eta} \hat{u} - \inf_{\mathbb{V}_\eta} \hat{u} \leq \kappa(g, \eta)\sigma^{-1}|\mathbb{T}^3|_g^{\frac{1}{2}}\|du\|_{L^2}.$$

Sup Bound

Proof.

- Observe that $\int_{\mathbb{V}_\eta} |d\hat{u}| \mathrm{vol}_{\pi^* g} \leq \kappa(g,\eta) \int_{\mathbb{T}^3} |du| \mathrm{vol}_g$.
- Use the coarea formula to obtain

$$\int_{\mathbb{V}_\eta} |d\hat{u}| \mathrm{vol}_{\pi^* g} = \int_{\inf_{\mathbb{V}_\eta}}^{\sup_{\mathbb{V}_\eta}} \int_{\hat{u}^{-1}\{t\}} 1_{\mathbb{V}_\eta} \mathrm{Area}_{\pi^* g}.$$

- Each $\pi(\hat{u}^{-1}\{t\})$ must contain a non-trivial element of $H_2(\mathbb{T}^3;\mathbb{Z})_{\mathbb{R}}$ as a subset: we have $\mathcal{H}^2(\hat{u}^{-1}\{t\}) \geq \sigma$.
- We obtain

$$\sup_{\mathbb{V}_\eta} \hat{u} - \inf_{\mathbb{V}_\eta} \hat{u} \leq \sigma^{-1} |\mathbb{T}|^{\frac{1}{2}} \|du\|_{L^2}.$$

$\square$

L^3 Bound

Conclusion

Putting everything together gives us

$$\|du\|_{L^3}^3 \leq \kappa(g,\eta)\sigma^{-1}|\mathbb{T}^3|^{\frac{1}{2}}\|du\|_{L^2}\left(\int_{\mathbb{T}^3}\frac{|\nabla du|^2}{|du|}\mathrm{vol}_g\right)^{\frac{1}{2}}\|du\|_{L^3}^{\frac{3}{2}}$$
$$+ 2\kappa(g,\eta)\sigma^{-1}\eta^{-1}|\mathbb{T}^3|^{\frac{1}{2}}\|du\|_{L^2}^3$$

Lemma

Let g be a Riemannian metric on $\mathbb{T}^3$ and let $\eta > 0$, $\sigma > 0$, $V > 0$, and $m \in \mathbb{N}$ be fixed constants. Suppose that $\min\{\mathrm{stabsys}_1(\mathbb{T}^3, g), \mathrm{stabsys}_2(\mathbb{T}^3, g)\} \geq \sigma$, that $\kappa(g, \eta) \leq m$, and $V^{-1} \leq |\mathbb{T}^3|_g \leq V$. Then, there is a constant $C(V, \sigma, m)$ and a map $\mathbb{U} : (\mathbb{T}^3, g) \to \mathbb{T}^3$ such that if u^i denote the components of $\mathbb{U}$, then du^i are harmonic one-forms, the cohomology classes $[du^i]$ form a basis of $H^1(\mathbb{T}^3; \mathbb{Z})_{\mathbb{R}}$, and for $i = 1, 2, 3$ we have

$$\max\{\|du^i\|_{L^2}, \|du^i\|_{L^3}\} \leq C(V, \sigma, \eta, m).$$

Lemma

Let $\eta, \sigma, V > 0$ *and* $m \in \mathbb{N}$. *There is a constant* $C(\eta, \sigma, V, m)$ *such that if* g *is a Riemannian metric on* $\mathbb{T}^3$ *with*

- $\min\{\mathrm{stabsys}_1(\mathbb{T}^3, g); \mathrm{stabsys}_2(\mathbb{T}^3, g)\} \geq \sigma$;
- $V^{-1} \leq |\mathbb{T}^3|_g \leq V$
- $\kappa(g, \eta) \leq m$,

then there is a map $\mathbb{U} : \mathbb{T}^3 \to \mathbb{T}^3$ *whose differentials are harmonic one-forms generating* $H^1(\mathbb{T}^3; \mathbb{Z})_{\mathbb{R}}$ *and satisfy*

- $\max\{\|du^i\|_{L^2}, \|du^i\|_{L^3}\} \leq C(V, \sigma, \eta, m)$.
- $\|dg_{ij}\|_{L^1} \leq C(V, \sigma, \eta, m)\|S_g^-\|_{L^2}^{\frac{1}{2}}$.