Gromov norm on nonpositively curved manifolds

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Gromov norm

- M is a connected, closed, oriented topological manifold,
- $\alpha \in H_k(M, \mathbb{R})$ is a singular homology class,

Definition (Gromov, Thurston)

The Gromov norm of $\alpha \in H_k(M, \mathbb{R})$ is given by

$$||\alpha||_1 := \inf\{\sum_{i=1}^{\ell} |a_i| \ : \ \sum_{i=1}^{\ell} a_i \sigma_i \text{ is a cycle representing } \alpha \text{ in } H_k(M, \mathbb{R})\}.$$

In particular, the Gromov norm of the fundamental class $[M]$ is called the simplicial volume of M, denoted by $||M||$.

Properties

- 1-classes have zero Gromov norm, $||S^1|| = 0$.
- Gromov norm is a semi-norm, $\forall \alpha, \beta \in H_k(M, \mathbb{R})$, $\lambda \in \mathbb{R}$,

$$||\alpha + \beta||_1 \leq ||\alpha||_1 + ||\beta||_1,$$

$$||\lambda\alpha||_1 = |\lambda| \cdot ||\alpha||_1.$$

- Functoriality: $\forall f : M \to N$, $\forall \alpha \in H_k(M, \mathbb{R})$,

$$||f_*(\alpha)||_1 \leq ||\alpha||_1.$$

- If $\dim(M) = \dim(N) \geq 3$, then $||M \# N|| = ||M|| + ||N||$.
- $||M|| \cdot ||N|| \leq ||M \times N|| \leq \binom{m+n}{m} ||M|| \cdot ||N||$ where m, n are the dimensions of M and N.

Examples: dim 1

- $||S^1|| = 0$.

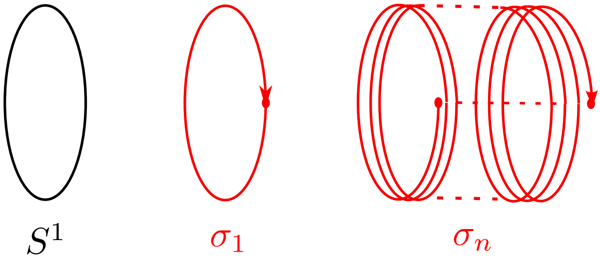
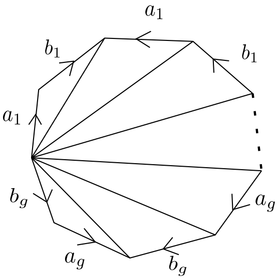


Figure: $||S^1|| = 0$

- $[S^1] = [1 \cdot \sigma_1]$, the L^1-norm is 1.
- $[S^1] = [\frac{1}{n} \cdot \sigma_n]$, the L^1-norm is $1/n$, which tends to 0.

Examples: dim 2

- $||S^2|| = 0$. $||T^2|| = ||S^1 \times S^1|| \leq 2 \cdot ||S^1|| \cdot ||S^1|| = 0$.
- $||\Sigma_g|| = 2|\chi(\Sigma_g)| = 4g - 4$ if $g \geq 2$.



$$[\Sigma_g] = [\sigma_1 + \sigma_2 + ... + \sigma_{4g-2}],$$

$$||\Sigma_g|| \leq 4g - 2.$$

Examples: Σ_g

- Improve the bound by **passing to finite covers**:
  - Take a degree d cover $f : \Sigma_h \to \Sigma_g$, where $(h-1) = d(g-1)$.
  - $f_*([\Sigma_h]) = d \cdot [\Sigma_g]$.
  - Use functoriality, $d \cdot ||\Sigma_g|| = ||f_*([\Sigma_h])||_1 \leq ||[\Sigma_h]||_1 = ||\Sigma_h||$.
  - Use $||\Sigma_h|| \leq 4h-2$ shown in the previous slide, we obtain,

$$||\Sigma_g|| \leq \frac{(4h-2)}{d} = \frac{(g-1)(4h-2)}{h-1} \to 4(g-1), \text{ as } d \to \infty.$$

  - Therefore,

$$||\Sigma_g|| \leq 4g-4 = 2|\chi(\Sigma_g)|.$$

- Use straightening method, one can show the other direction of inequality

$$||\Sigma_g|| \geq 4g-4.$$

Straightening

For any $[\Sigma_g] = [\sum_i a_i \sigma_i]$, we have

$$\begin{aligned} 2\pi(2g-2) = \mathrm{Vol}(\Sigma_g) = \int_{\Sigma_g} dV &= \int_{[\sum_i a_i \cdot \sigma_i]} dV \\ &= \int_{[\sum_i a_i \cdot st(\sigma_i)]} dV \quad (Straightening) \\ &\leq \sum_i |a_i| \cdot |\operatorname{Vol}(st(\sigma_i))| \\ &\leq \left(\sum_i |a_i| \right) \cdot \pi \end{aligned}$$

Take infimum, we obtain $||\Sigma_g|| \geq 4g - 4$.

Examples: dim 3

- Use Thurston's geometrization decomposition,

$$M = M_1 \# M_2 \# ... \# M_k,$$

$$M_i = N_{i_1} \cup_{T^2} N_{i_2} \cup_{T^2} ... \cup_{T^2} N_{i_s},$$

each N_{i_j} belongs to one of the 8 geometries

$$S^3,\ \mathbb{R}^3,\ \mathbb{H}^3,\ S^2 \times \mathbb{R},\ \mathbb{H}^2 \times \mathbb{R},\ \widetilde{\mathrm{SL}_2(\mathbb{R})},\ \mathrm{Nil},\ \mathrm{Sol}.$$

Theorem (Gromov '82, Thurston '77)

$$||M|| = \frac{\mathrm{Vol}(\text{hyperbolic pieces of } M)}{\sigma_3},$$

where σ_3 is the maximal volume of ideal simplices in $\mathbb{H}^3$.

Examples: dim 3

Theorem (Gromov '82, Thurston '77)

If M^n admits a hyperbolic metric, then

$$||M|| = \frac{\mathrm{Vol}_h(M)}{\sigma_n}.$$

Theorem (Gabai '83)

If M is a closed 3-manifold, then for any $\alpha \in H_2(M, \mathbb{R})$,

$$||\alpha||_1 = 2 \cdot ||\alpha||_{Th}.$$

Definition

$$||\alpha||_{Th} = \min_S \{\chi_-(S) : S \subset M \text{ represents } \alpha\},$$

where $\chi_-(S) = \max\{0, -\chi(S)\}$, and $\alpha \in H_2(M, \mathbb{Z})$.

More examples

Theorem (Trauber, Gromov '82)

If $\pi_1(M)$ is amenable, then $||\alpha||_1 = 0$ for all $\alpha \in H_k(M, \mathbb{R})$.

Finite groups, abelian groups and solvable groups are amenable, so the conclusion includes manifolds of non-negative Ricci curvatures.

Theorem (Inoue-Yano '82)

If M admits a negatively curved metric, then $||\alpha||_1 > 0$ for all nonzero $\alpha \in H_k(M, \mathbb{R})$ with $k \geq 2$.

Theorem (Mineyev '01)

If M is aspherical and $\pi_1(M)$ is Gromov hyperbolic, then $||\alpha||_1 > 0$ for all nonzero $\alpha \in H_k(M, \mathbb{R})$ with $k \geq 2$.

Relating to other invariants

- **Curvature**:
  - If $||M|| = 0$, then M admits no negatively curved metric.
  - If $||M|| > 0$, then M admits no (Ricci) positively curved metric.

Question

What about non-positively curved metric?

- **Mapping degree**: If $f : M \to N$ is a continuous map, then

$$|\deg(f)| \cdot ||N|| \leq ||M||.$$

In particular, if $||M|| > 0$, then any continuous self map has degree 0 or ± 1.

Relating to other invariants

- **Minimal volume/entropy**:

$$||M|| \leq C(n)\,\mathrm{Minvol}(M), \quad ||M|| \leq C(n)\,\mathrm{Minent}(M)^n.$$

Definition

$$\mathrm{Minvol}(M) := \inf_g \{\mathrm{Vol}_g(M) : |K_g| \leq 1\}$$

$$\mathrm{Minent}(M) := \inf_g \{h_{top}(M, g) : \mathrm{Vol}_g(M) = 1\}$$

- **Euler characteristic/L^2-betti numbers**:

Conjecture (Gromov)

If M is aspherical, and $||M|| = 0$, then $\chi(M) = 0$.

Manifolds of non-positive curvature

Question

Given a closed non-positively curved manifold M, and a homology class $\alpha \in H_k(M, \mathbb{R})$, is $||\alpha||_1 = 0$ or $||\alpha||_1 > 0$?

- If M is hyperbolic, then $||M|| > 0$.
- If $M = N \times S^1$, then $||M|| = 0$.

Conjecture (Gromov)

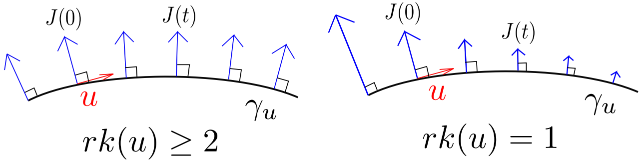
If M has (sectional curvature) $K \leq 0$ and $\mathrm{Ric} < 0$, then $||M|| > 0$.

- True in dimension 2, 3.
- Open in general.

Classification of closed non-positively curved manifolds

Definition

Let M be a nonpositively curved manifold. A unit tangent vector $u \in T^1M$ is called **geometric rank one** if there is no orthogonal parallel Jacobi field along the geodesic tangent to u. M is called **geometric rank one** if M has at least one geometric rank one vector.



Rank rigidity theorem

Theorem (Ballmann-Brin-Burns-Eberlein-Spatzier '85-'87)

If M is closed and non-positively curved, then either

1. M is geometric rank one, or
2. M is locally a (Riemannian) product, or
3. M is locally symmetric.

Gromov norm:

$$K \leq 0 \begin{cases} \text{Geometric rank one} \begin{cases} K < 0(\checkmark) \\ \text{Presence of zero sectional curvature}(*) \end{cases} \\ \text{Higher rank} \begin{cases} \text{Locally a product}(\checkmark) \\ \text{Locally symmetric}(**) \end{cases} \end{cases}$$

Geometric rank one manifolds*

Examples: (Generalized) graph manifolds

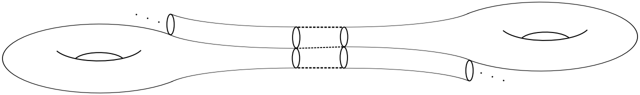


Figure: $||M|| = 0$

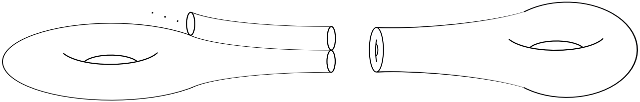


Figure: $||M|| > 0$

Theorem (Connell-W '20)

If M has non-positive curvature and there exists a point $x \in M$ such that every tangent vector at x is rank$^+$ one, then $||M|| > 0$.

Corollary

If M has non-positive curvature and a point of negative curvature, then $||M|| > 0$.

Corollary (Gromov's conjecture in dimension 3)

If M^3 has non-positive curvature and negative Ricci curvature, then $||M|| > 0$.

- Use [Chapter 7, Exercise 6(b), Petersen–Riemannian geometry]

$$\int_M Ric(X,X)dV = \int_M (\mathrm{div}\, X)^2 - \mathrm{tr}(\nabla X)^2 dV.$$

(Locally) symmetric spaces**

- Let G be a connected, finite center, semisimple Lie group of non-compact type, $K < G$ be a maximal compact subgroup, $X = G/K$ is the associated symmetric space.
- Let $\Gamma < G$ be a torsion-free, cocompact lattice, then $M = \Gamma \backslash G/K$ is a closed locally symmetric manifold.
- Take $G = \mathrm{SL}_n(\mathbb{R})$, $K = \mathrm{SO}(n)$, Γ a torsion-free cocompact lattice,

$$M = \Gamma \backslash \mathrm{SL}_n(\mathbb{R}) / \mathrm{SO}(n).$$

- With the natural G-invariant metric (induced by the Killing form), M is non-positively curved.
-

$$\mathrm{rank}(M) := \max\{r : \exists \mathbb{R}^r \to M \text{ totally geodesic immersion}\}$$

Gromov norm on locally symmetric manifolds

$M = \Gamma\backslash G/K$ is a closed locally symmetric manifold, $\alpha \in H_k(M, \mathbb{R})$ is a non-trivial class.

Theorem (Lafont-Schmidt '06)

$$||M|| > 0.$$

Theorem (Lafont-W '19)

If $k \geq \dim(M) - \mathrm{rank}(M) + 2$, then $||\alpha||_1 > 0$ for any non-trivial class.

Theorem (W '22)

There exists a cocompact lattice $\Gamma < \mathrm{SL}_n(\mathbb{R})$, and a non-trivial homology class $\alpha \in H_k(M, \mathbb{R})$ where $k = \dim(M) - \mathrm{rank}(M)$, such that $||\alpha||_1 = 0$.

Why in degree $\dim(M) - \operatorname{rank}(M)$?

For higher rank symmetric space $X_n = G/K = \mathrm{SL}_n(\mathbb{R})/\,\mathrm{SO}(n)$.

- There is a Lie subgroup $G_0 < G$, where

$$G_0 = \left(\begin{array}{c|c} \frac{1}{t^{(n-1)}} \cdot \mathrm{SL}_{n-1}(\mathbb{R}) & 0 \\ \hline 0 & t \end{array} \right), t > 0$$

$$\cong \mathrm{SL}_{n-1}(\mathbb{R}) \times \mathbb{R}$$

- This corresponds to a totally geodesic submanifold $X_{n-1} \times \mathbb{R} \subset X_n$, where

$$X_{n-1} \times \mathbb{R} = \mathrm{SL}_{n-1}(\mathbb{R})/\,\mathrm{SO}(n-1) \times \mathbb{R}$$

- $\dim(X_{n-1} \times \mathbb{R}) = \dim(X_n) - \operatorname{rank}(X_n)$.
- Our class $\alpha \in H_k(M)$ constructed is covered by this $X_{n-1} \times \mathbb{R}$.

More general statement

Theorem (W '20)

If $k \geq srk(\tilde{M}) + 2$, then $||\alpha||_1 > 0$ for any non-trivial class.

Definition

Let X be a symmetric space of non-compact type, the splitting rank is defined as

$$srk(X) := \max\{\dim(Y \times \mathbb{R}) : Y \times \mathbb{R} \to X$$

$$\text{is a totally geodesic embedding }\}$$

- If $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, then $srk(X) = \dim(X) - \mathrm{rank}(X)$.
- For any symmetric space X, $srk(X) \leq \dim(X) - \mathrm{rank}(X)$, and "$=$" holds if and only if X is of type $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$.

Barycentric straightening

Let $\{V_0, ..., V_k\}$ be the set of vertices of a given k-simplex σ.

- Associate each V_i a finite measure μ_i supports on $\partial_\infty \widetilde{M}$.
- Fill a simplex in the space of finite measures.

$$\{\sum_{i=0}^{k} a_i \mu_i : 0 \leq a_i \leq 1, \sum_{i=0}^{k} a_i = 1\}$$

- Take the barycenter map.

$$\mu \mapsto \text{ the unique point } x \text{ such that } \int_{\partial_\infty \widetilde{M}} v_{x\theta} d\mu(\theta) = 0$$

- **Upshot: the Jacobian of the barycentrically straightened simplices are uniformly bounded.**

Thank You!