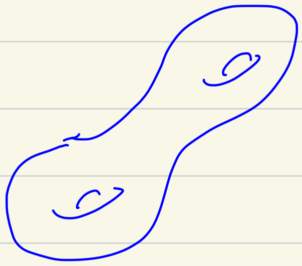
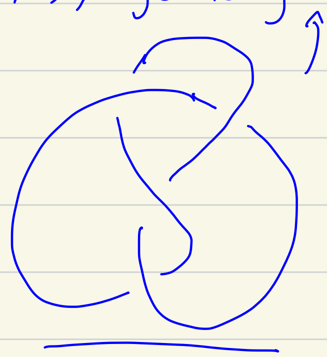
Geometry & Topology at Lanzhou University

Thurston norm and the geometry of surfaces

Mostow rigidity & effective geometrization.

M^n finite-volume, hyperbolic, $n \geq 3$, geometry determined by π_1.





Riley 72.

Quantitative topology & geometry.

$$\phi \in H^1(M^3;\mathbb{Z}) \cong H_2(M;\mathbb{Z}) \ni S$$

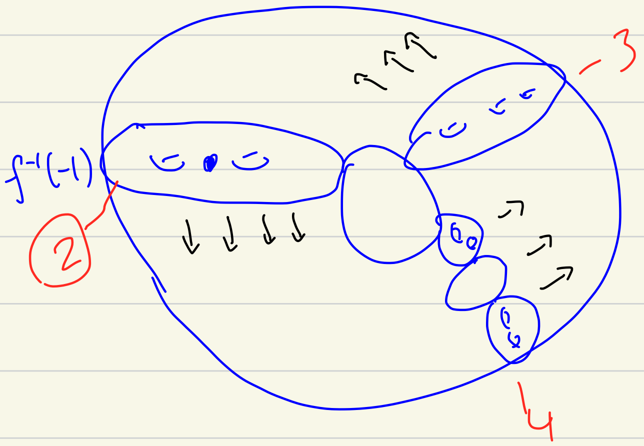
Thurston norm

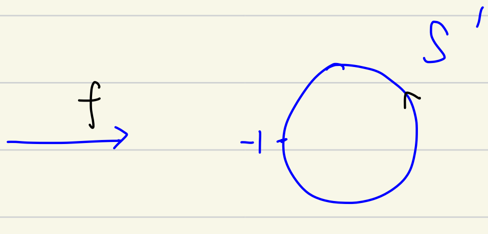
M^3 compact irreducible, the Thurston norm on ϕ

$$\chi_-(S) = \max\{0, -\chi(S)\}$$

$$\|\phi\|_{Th} = \min\{\chi_-(S) \mid S \text{ is embedded dual to } \phi\}.$$

$\phi \in H^1(M^n;\mathbb{Z})$, $f: M^n \longrightarrow S^1$. ϕ locally df

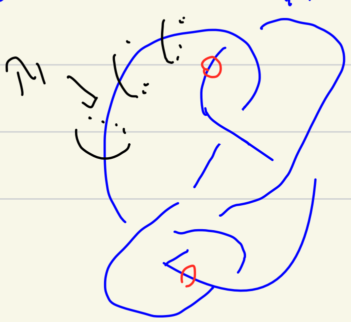


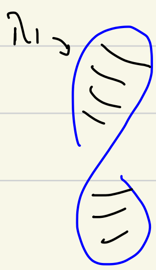


in this example

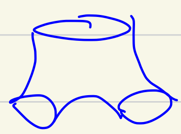
$$\|\phi\|_{Th} = |2-2g| = 2.$$

M is hyperbolic, $\| \ \|_{Th}$ genuine. $\|\alpha\| \neq 0 \Rightarrow [\alpha] \neq 0 \in H^1(M;\mathbb{Z})$.





2-punctured disk



$\chi = -1$

$$\|\lambda_1^{\#}\|_{Th} = 1$$

L^2- norm

$\phi \in H^1$. $\rightsquigarrow$ α is a 1-form metric.

$$|\alpha|^2_{L^2} = \int_M \underline{|\alpha(x)|^2} dV. = \int_M \alpha \wedge * \alpha.$$

$$\phi \in H^1(M;\mathbb{R}), \quad \|\phi\|_{L^2} = \inf \{ |\alpha|_{L^2} \mid \underline{\alpha \sim \phi} \}$$

Kronheimer- Mrowka '97.

$$\|\phi\|_{Th} = \inf_{\substack{h \\ \uparrow \\ \text{metric}}} \frac{1}{4\pi} \underline{\|\phi\|_{L^2(h)}} \, \underline{\|s_h\|_{L^2(h)}}$$

$s_h = 0$

$\| \, \|_{Th} = 0$ $\quad (\pi_1)\,ab$

$$\mathrm{vol}(M) = 6\pi \lim_{n\to\infty} \frac{\log |(H_1(M;\mathbb{Z}))_{tor}|}{n}$$

Le.

Bergeron - Sengün- Venkatesh '15

M_n

$\downarrow$ cover finite deg n

M_0

$\uparrow$ fix 3-dim hyperbolic

$$\frac{C_1}{\mathrm{Vol}(M)} \| \cdot \|_{Th} \le \| \cdot \|_{L^2} \le C_2 \| \cdot \|_{Th} \quad H^1(M;\mathbb{R}).$$

C_1, C_2 depend on M_0.

Brock - Dunfield '17

For all closed orientable hyperbolic 3-mf,

$$\frac{\pi}{\sqrt{\mathrm{vol}(M)}} \| \, \|_{Th} \le \| \, \|_{L^2} \le \frac{10\pi}{\sqrt{\mathrm{Inj}}} \| \, \|_{Th} \quad H^1(M;\mathbb{R})$$

99.99%

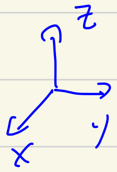
Stern '19, Bray- Stern '19.

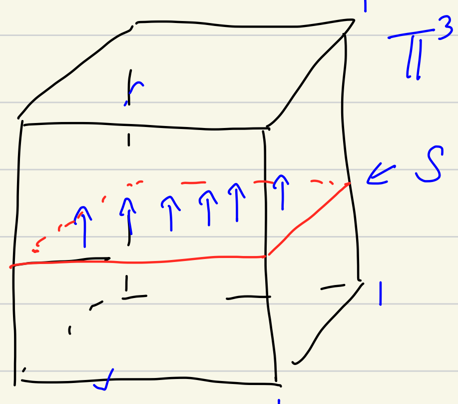
3-mf with boundary, reducible.

$$dz \in H^1(\mathbb{T}^3;\mathbb{Z})$$

dual to $S = \mathbb{T}^2$.

$\chi(\mathbb{T}^2) = 0$. $\quad \|dz\|_{Th} = 0$.





$$\|dz\|^2_{L^2} = \int_M dz \wedge * dz = \int_M dx dy dz = 1.$$

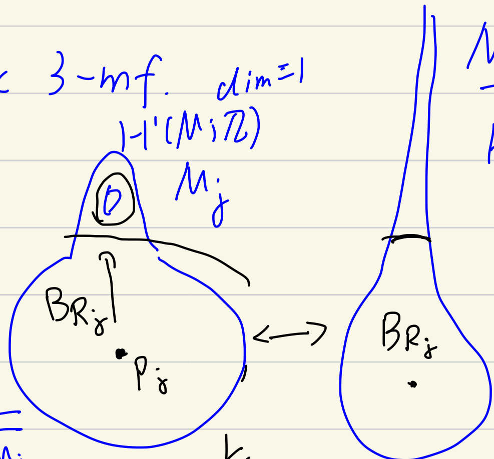
Conjecture of BD.

M_j orientable, closed hyperbolic 3-mf. dim=1 $H^1(M_j;\mathbb{Z})$

$M_j \longrightarrow M$

Gromov-Hausdorff.

$M = S^3 - K$. $H^1(M;\mathbb{Z})$. 2-dim



$$\sup_{H^1(M_j)} \frac{\|\ \|_{L^2}}{\|\ \|_{Th}} \overset{\leq}{\sim} O\left(\sqrt{\log \frac{1}{\mathrm{inj} M_j}}\right). \ll \frac{1}{\sqrt{\mathrm{inj} M_j}}$$

k_j - quasi-isometric

$k_j \to 1$, $R_j \to \infty$

global $\mathrm{vol}(M) < \infty$.

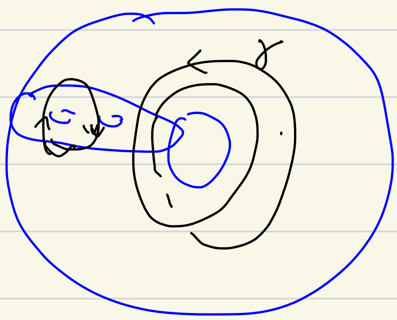
$\mathrm{vol}(M_j) < \mathrm{vol}(M). < \infty$

uniform upper bound.

$\phi \in H^1(M;\mathbb{Z})$, geodesic γ.

$\int_\gamma \phi$ = algebraic intersection between γ and S (dual to γ).

for this example. $= 2$



$\|\phi\|_{Th} = \min \{ |\chi(S)| \mid S \text{ dual to } \phi \}$.

$\|\phi\|_{Th} \leq ?$ find a S dual to ϕ.

H. '23. M has n short geodesic, $\ell(\gamma) \leq 0.1$

$$\|\phi\|_{Th} \geq \sum_{i=1}^{n} \underbrace{\left|\int_{\gamma_i} \phi\right|}_{S \,\backslash\!\backslash\, 3} \underbrace{\left(\frac{0.107}{\sqrt{\ell(\gamma_i)}} - 1\right)}.$$

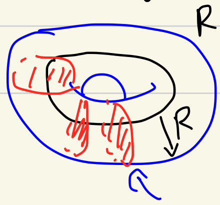
γ short.:

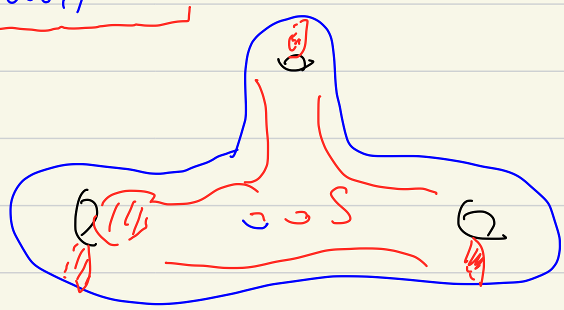
embedded nbhd.

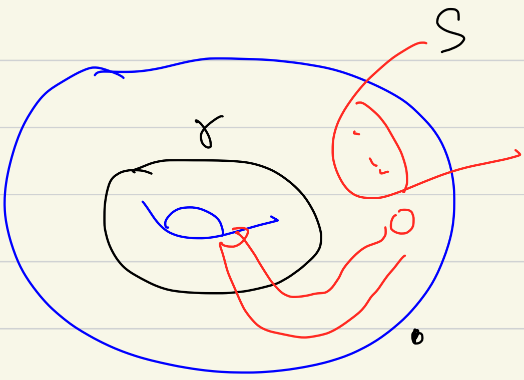
$S^1 \times D^2$

γ short

R large.





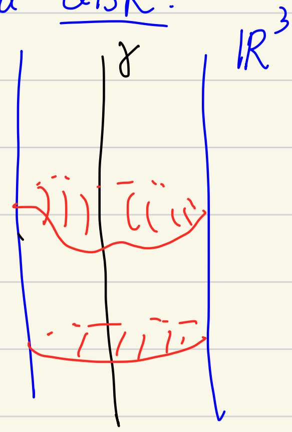


$\int_\gamma \phi \not\equiv 0$,

$\Rightarrow$ $S \cap$ nbhd of γ

contains an essential disk.

S least area



S totally geodesic,

disk area minimizes.

Corollary sharp. for many M_j $\mathrm{inj}\, M_j \to 0$

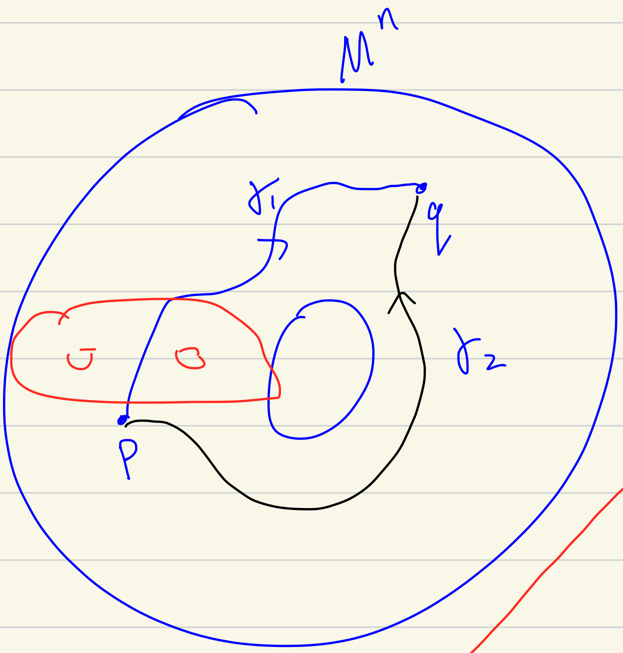
$$\|\ \|_{Th} \sim O\left(\frac{1}{\sqrt{\ell(\gamma)}}\right)$$

H: 23

Let $V>0$. M^3: $vol(M) < V$. $\exists\, c(V)$ s.t. $\forall M$: $vol(M) < V$,

$$\frac{\|\ \|_{L^2}}{\|\ \|_{Th}} \leq 2c(V)\pi + 10\sqrt{\pi \log\left(\frac{V/\pi}{\mathrm{inj}} + 1\right)}.$$

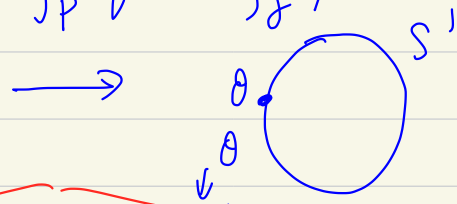
□.



$\phi \in H^1(M;\mathbb{Z}) \cong [M : S^1]$

$f : M \to S^1$ circle-valued map.

$f_p(q) = \int_\gamma \phi$ $: M \to \mathbb{R}/\mathbb{Z}$



$f : M \to S^1$.

· compact compact

Thurston Norm paper

homotopic

$\exists\, f_1$ smooth $f_1 \in [f]$

$f_1^* d\theta = \phi$.

Regular value theorem, $\exists$ a regular value θ.

$f^{-1}(\theta) \subseteq M$ smooth embedded submanifold.

codim -1

$\mathbb{Q} \to \mathbb{R}$

· $\|\ \|_{Th}$ continuous