Riemannian manifolds and the Sobolev-Neumann constants

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Riemannian Manifolds

Definition

Let M be a smooth manifold. Then, a smooth section of the vector bundle $T^*M \otimes T^*M$, say g, such that

1. for all $p \in M$ and $v \in T_pM$ we have $g_p(v, v) \geq 0$, with equality if and only if $v = 0$;
2. for all $p \in M$ and $v, w \in T_pM$ we have $g_p(v, w) = g_p(w, v)$,

then we call g a Riemannian metric on M, and call the tuple (M, g) a Riemannian manifold.

Metrics Exist

Lemma

Let M be a smooth manifold. Then there is always at least one Riemannian metric on M; actually there are in general many.

Proof.

1. Locally we can pull back the Riemannian metric arising from coordinates.
2. Stitch these locally define metrics together using a partition of unity.

□

Connections on manifolds

Definition

Let M be a smooth manifold. A Koszul connection, say ∇, is a bi-linear map $\nabla : \Gamma(TM) \times (TM) \to \Gamma(TM)$ which satisfies the following two additional properties: for any $X, Y \in \Gamma(TM)$ and smooth function $f : M \to \mathbb{R}$ we have

1. $\nabla_{fX} Y = f \nabla_X Y$,
2. and $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

Fundamental Theorem

Theorem

Let (M, g) be a Riemannian manifold. Then there exists a unique Koszul connection on M, say ∇, which for all $X, Y, Z \in \Gamma(X)$ satisfies

1. $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,
2. *and* $\nabla_X Y - \nabla_Y X = [X, Y]$.

This unique connection is called the Levi-Civita connection on the Riemannian manifold (M, g).

Gradients of functions

Definition

Let (M, g) be a Riemannian manifold, and let $f : M \to \mathbb{R}$ be a smooth function on M. Given any $p \in M$ we define the gradient of f at p, denoted $(\nabla f)_p$ to be the unique vector field such that for all $w \in T_pM$ we have

$$g_p(w, (\nabla f)_p) = df_p(w). \tag{1}$$

This defines a smooth vectorfield $p \mapsto (\nabla f)_p$, which is called the gradient field, or just gradient, of f.

Volume and Area

Definition

Let M be an oriented smooth manifold, and let g be Riemannian metric on M. Then, we may get a unique non-vanishing top-dimensional form on M as follows. For each p, let $\{E_i\}_1^m$ be a positively oriented orthonormal basis of T_pM with respect to g_p, and set θ^i to be the dual covector of E_i, that is we have $\theta^i(E_j) = \delta^i_j$, the Kronecker delta. Set

$$\mathrm{Vol}_g = \theta^1 \wedge \cdots \wedge \theta^m.$$

Volume and Area

Definition

Let (M, g) be an oriented smooth $m - \textit{dimensional}$ Riemannian manifold, and let Σ be a smooth $m - 1$ dimensional embedded submanifold of M. If the orientation on M induces and orientation on Σ, then we call Σ and oriented surface, or a two-sided surface. In this case, $g|_{\Sigma}$ defines a unique $m - 1$ form on Σ, and we will denote it by A_{Σ}, or just A.

Volume and Area

Lemma

Let (M, g) be an oriented smooth $m-$ dimensional Riemannian manifold, and suppose $\Sigma \subset M$ is an embedded $m-1$-dimensional submanifold of M such that there is an open region $\Omega \subset M$ with the property that $\partial\Omega = \Sigma$. Then, the surface Σ is oriented.

Proof.

There exists a vectorfield ν on $\partial\Omega$ called the inward pointing normal. This orients $\partial\Omega = \Sigma$. □

Volume and Area

Definition

Let (M, g) be an oriented smooth $m - \textit{dimensional}$ Riemannian manifold, and let $\Sigma \subset M$ be an embedded oriented $m - 1$ dimensional surface. Then, we set $\mathrm{Area}(\Sigma) = \int_\Sigma 1 \mathrm{A}_\Sigma$.

The $W^{1,1}$-norm

Definition

Let (M, g) be a Riemannian manifold, and let $f : M \to \mathbb{R}$ be a smooth function on M. Then, we set

$$\|f\|_{W^{1,1}(M,g)} = \int_M \sqrt{g(\nabla f, \nabla f)} + |f| \mathrm{Vol}. \quad (2)$$

Set $W^{1,1}(M, g)$ to be the closure of $C^\infty(M)$ under the norm $\| \cdot \|_{W^{1,1}(M,g)}$.

The Sobolev-Neumann constants

Definition

Let (M, g) be an m-dimensional Riemannian manifold, let ∇ be its Levi-Civita connection, and let $\alpha \in \left[1, \frac{m}{m-1}\right]$. Then, we define $SN_\alpha(M, g)$ to be the following constant:

$$SN_\alpha(M, g) = \inf \left\{ \frac{\int_M \sqrt{g(\nabla f, \nabla f)}}{\min_{k \in \mathbb{R}} \left(\int_M |f - k|^\alpha \right)^{\frac{1}{\alpha}}} : f \in W^{1,1}(M, g) \right\}$$

The Isoperimetric-Neumann constants

Definition

Let (M, g) be a smooth oriented m dimensional Riemannian manifold, and let $\alpha \in [1, \frac{m}{m-1}]$. We define $IN_\alpha(M, g)$ as follows:

$$IN_\alpha(M, g) = \inf \left\{ \frac{\mathrm{Area}(\partial\Omega)}{\min\left\{ |\Omega|^{\frac{1}{\alpha}}, |\Omega^c|^{\frac{1}{\alpha}} \right\}} : \Omega \subset\subset M \right\}$$

SN vs IN

Lemma

Let (M, g) be a smooth oriented m dimensional Riemannian manifold. Then, we have that $SN_1(M, g) = IN_1(M, g)$.

Proof.

1. One can show that for each $f \in C^\infty$ with $\|f\|_{W^{1,1}(M,g)} < \infty$; there is a $k \in \mathbb{R}$ such that $M_+(k) = \{f > k\}$ and $M_-(k) = \{f < k\}$ satisfy $\max\{|M_+(k)|, |M_-(k)|\} \leq \frac{|M|}{2}$;
2. $\int_{M_+(k)} |\nabla f| = \int_0^\infty \mathrm{Area}((f-k)^{-1}\{t\})dt$;
3. For $t > 0$ we have $\{(f-k) > t\} \subset M_+(k)$, and $(f-k)^{-1}\{t\} = \partial\{(f-k) > t\}$;
4. $\int_0^\infty \mathrm{Area}(f-k)^{-1}\{t\}dt \geq IN_1 \int_0^\infty |\{(f-k) > t\}|dt$ which in turn is $\int_{\{(f-k)>0\}}(f-k)$.

□