Rational ellipticity of Riemannian manifolds

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Introduction

Definition: The n-th homotopy group of a topological space X is the set of homotopy classes of maps from the n-sphere to X, with a group structure, and is denoted $\pi_n(X)$.

Theorem (Serre)
1. The homotopy groups of an odd dimensional sphere S^n are torsion except in dimension n;
2. The homotopy groups of an even dimensional sphere S^n are torsion except in dimension n and $2n-1$.

We look at $\pi_i(X) \bigotimes \mathbb{Q}$. There is a big theory to study it using differential forms, e.g, Sullivan's miminal model theory.

relate rational homotopy group to geometry

Conjecture 1 (Bott-Grove-Halperin): A compact simply connected Riemannian manifold M with nonnegative sectional curvature is rationally elliptic.

Conjecture 2 (Paternain): A compact simply connected Riemannian manifold M with completely integrable geodesic flow is rationally elliptic.

Rational ellipticity

A simply connected manifold M is said to be rationally elliptic if and only if it has finite dimensional rational homotopy groups, i.e., all but finitely many homotopy groups of M are finite.

Examples of rational elliptic manifolds: 1. Compact simply connected homogeneous spaces and cohomogeneity one manifolds.

2. $M = D(B_1) \cup_E D(B_0)$ is rationally elliptic if and only if E is. (Grove-Halperin)

Properties of rational elliptic manifolds:

1. M being rationally elliptic is equivalent to polynomial growth of the sequence of Betti numbers of its based loop space ΩM, i.e, $\sum_{i=0}^{k-1} dimH_i(\Omega M)$ grows polynomially when $k \to \infty$. (by Sullivan's miminal model theory)

2. M has nonnegative Euler characteristic number; (Felix-Halperin-Thomas)

3. $dimH_*(M, \mathbb{Q}) \leq 2^n$. (Felix-Halperin-Thomas)

4. If $dimM = 4$, then M is homeomorphic to $\mathbb{S}^4, \mathbb{CP}^2, \mathbb{S}^2 \times \mathbb{S}^2, \mathbb{CP}^2 \pm \mathbb{CP}^2$. (Paternain)

Progress

Theorem (Grove, Wilking, Yeager, 2019): Bott-Grove-Halperin's conjecture is true when M supports an isometric action with principal orbits of codimension two.

Remark: Based on an iterated use of the Rauch comparison theorem for Jacobi fields, an estimate for the Betti numbers of ΩM for manifolds with $0 < \delta \leq secM \leq 1$ was derived by Berger and Bott. Although the estimate is given in terms of the pinching constant δ, its growth rate is exponential.

Theorem (Chen) : Let (M, g) be a n-dimensional compact simply connected real analytic Riemannian manifold that has entire Grauert tube, then M is rationally elliptic.

Here (M, g) is said to be real analytic if M is a real analytic manifold with a real analytic Riemannian metric g. Then there is a unique adapted complex structure defined on $T^R M = \{v \in TM | g(v, v) < R^2\}$ for some $R > 0$. When $R = \infty$, then M is said to have entire Grauert tube. It was shown by Lempert and Szöke that a Riemannian manifold with entire Grauert tube has nonnegative sectional curvature.

All known manifolds with entire Grauert tube are obtained by Aguilars construction: starting with a compact Lie group with a bi-invariant metric, or the product of such a group with Euclidean space, one takes the quotient by some group of isometries acting freely.

It was conjectured by Hopf that the Euler characteristic number of a compact Riemannian manifold with nonnegative sectional curvature is nonnegative. The following corollary settles this conjecture under the stronger assumption that M has entire Grauert tube.

Corollary: Let M be a compact Riemannian manifold with entire Grauert tube. Then M has nonnegative Euler characteristic number.

Proof: If M has finite fundamental group, then its universal cover $\widetilde{M}$ with the induced Riemannian metric also has entire Grauert tube. Then $\widetilde{M}$ is rationally elliptic. Hence the Euler characteristic number of $\widetilde{M}$ is nonnegative. Then M has nonnegative Euler characteristic number. If M has infinite fundamental group, as M has nonnegative sectional curvature, then the Euler characteristic number of M is zero.

A related conjecture proposed by Totaro predicts that a compact Riemannian manifold M with nonnegative sectional curvature has a good complexification, i.e., M is diffeomorphic to a smooth affine algebraic variety U over the real number such that the inclusion $U(\mathbb{R}) \to U(\mathbb{C})$ is a homotopy equivalence. The Euler characteristic number of a compact manifold which has a good complexification is also nonnegative. Also, a conjecture by Burns predicts that for every compact Riemannian manifold M with entire Grauert tube, the complex manifold TM is an affine algebraic variety in a natural way. If this is correct, the complex manifold TM would be a good complexification of M in the above sense. Both conjectures of Totaro and Burns are still open.

Approach via Morse theory

Let M be a n-dimensional compact manifold endowed with a Riemannian metric g. For $x \in M$ and each $T > 0$, let

$$D_T := \{v \in T_x M | g(v, v) \leq T^2\}$$

be the disk of radius T in $T_x M$. Define the counting function $n_T(x, y)$ by

$$n_T(x, y) := \sharp((exp_x)^{-1}(y) \cap D_T).$$

In other words, $n_T(x, y)$ counts the number of geodesic arcs joining x to y with length $\leq T$.

Theorem (Berger and Bott):

$$\int_M n_T(x,y)dy = \int_0^T d\sigma \int_{\mathbb{S}} \sqrt{det(g(J_j(\sigma), J_k(\sigma)))_{j,k=1,2,\cdots,n-1}}\, d\theta,$$

where $J_j, j = 1, 2, \cdots, n-1$ are Jacobi fields along the unique geodesic γ determined by $\theta \in \mathbb{S}$ with initial conditions

$$J_j(0) = 0$$

$$J_j'(0) = v_j.$$

Theorem (Gromov): Let M be a n-dimensional compact simply connected manifold endowed with a Riemannian metric g, then

$$\sum_{j=0}^{k-1} dimH_j(\Omega M) \leq \frac{1}{Vol_g(M)} \int_M n_{Ck}(x,y)dy.$$

Vertical and horizontal subbundles

Let $\pi : TM \to M$ be the canonical projection, i.e., if $\theta = (x, v) \in TM$, then $\pi(\theta) = x$. There exists a canonical subbundle of TTM called the vertical subbbundle whose fiber at θ is given by the tangent vectors of curves $\sigma : (-\epsilon, \epsilon) \to TM$ of the form: $\sigma(t) = (x, v + t\omega)$, where $\omega \in T_xM$. In other words,

$$V(\theta) = ker((\pi_*)_\theta).$$

Suppose that M is endowed with a Riemannian metric g. We shall define the connection map

$$K : TTM \to TM$$

as follows:

let $\xi \in T_\theta TM$ and $z : (-\epsilon, \epsilon) \to TM$ be an adapted curve to ξ, that is, with initial conditions as follows:

$$z(0) = \theta$$

$$z'(0) = \xi.$$

such a curve gives rise to a curve $\alpha : (-\epsilon, \epsilon) \to M, \alpha := \pi \circ z$ and a vector field Z along α, equivalently, $z(t) = (\alpha(t), Z(t))$.

Define

$$K_\theta(\xi) := (\nabla_\alpha Z)(0) = \lim_{t \to 0} \frac{(P_t)^{-1} Z(t) - Z(0)}{t},$$

where $P_t : T_x M \to T_{\alpha(t)} M$ is the linear isomorphism defined by the parallel transport along α. The horizontal subbundle is the subbundle of TTM whose fiber at θ is given by

$$H(\theta) = ker K_\theta.$$

Another equivalent way of constructing the horizontal subbundle is by means of the horizontal lift

$$L_\theta : T_xM \to T_\theta TM,$$

which is defined as follows: let $\theta = (x, v)$. Given $\omega \in T_xM$ and $\alpha : (-\epsilon, \epsilon) \to M$ an adapted curve of ω, i.e., $\alpha(0) = x, \alpha'(0) = \omega$. Let $Z(t)$ be the parallel transport of v along α and $\sigma : (-\epsilon, \epsilon) \to TM$ be the curve $\sigma(t) = (\alpha(t), Z(t))$. Then

$$L_\theta(w) = \sigma'(0) \in T_\theta TM.$$

K_θ and L_θ have the following properties:

$$(\pi_*)_\theta \circ L_\theta = Id$$

$$K_\theta \circ i_* = Id,$$

where $i : T_x M \to TM$ is the inclusion map. Moreover,

$$T_\theta TM = H(\theta) \oplus V(\theta)$$

and the map $j_\theta : T_\theta TM \to T_x M \times T_x M$ given by

$$j_\theta(\xi) = ((\pi_*)_\theta(\xi), K_\theta(\xi))$$

is a linear isomorphism.

For each $\theta \in TM$, there is a unique geodesic γ_θ in M with initial condition θ. Let $\xi \in T_\theta TM$ and $z : (-\epsilon, \epsilon) \to TM$ be an adapted curve to ξ, that is, with initial conditions as follows:

$$z(0) = \theta$$

$$z'(0) = \xi.$$

Then the map $(s, t) \mapsto \pi \circ \phi_t(z(s))$ gives rise to a variation of γ_θ. Here $\pi : TM \to M$ is the projection map and ϕ_t is the geodesic flow of TM. The curves $t \mapsto \pi \circ \phi_t(z(s))$ are geodesics and therefore the corresponding variational vector fields $J_\xi := \frac{\partial}{\partial s}|_{s=0} \pi \circ \phi_t(z(s))$ is a Jacobi field with initial conditions

$$J_\xi(0) = (\pi_*)_\theta(\xi)$$

$$J'_\xi(0) = K_\theta(\xi).$$

Adapted complex structure on the tangent bundle

For $\tau \in \mathbb{R}$ denote by $N_\tau : TM \to TM$ the smooth mapping defined by multiplication by τ in the fibers. If $\gamma : \mathbb{R} \to M$ is a geodesic, define an immersion $\phi_\gamma : \mathbb{C} \to TM$ by

$$\phi_\gamma(\sigma + i\tau) = N_\tau \gamma'(\sigma).$$

The images of $\mathbb{C} \setminus \mathbb{R}$ under the mapping ϕ_γ defines a smooth foliation of $TM \setminus M$ by surfaces. Moreover, each leaf has complex structure that it inherits from $\mathbb{C}$ via ϕ_γ.

Given $R > 0$, put

$$T^R M = \{v \in TM | g(v, v) < R^2\}.$$

A smooth complex structure on $T^R M$ will be called adapted if the leaves of the foliation with the complex structure inherited from $\mathbb{C}$ are complex submanifolds.

Theorem: (Lempert and Szöke, Guillemin and Stenzel) Let M be a compact real analytic manifold equipped with a real analytic metric g. Then there exists some $R > 0$ such that $T^R M$ carries a unique adapted complex structure.

When the adapted complex structure is defined on the whole tangent bundle, i.e. $R = \infty$, then M is said to have entire Grauert tube. It was shown by Lempert and Szöke that a Riemannian manifold with entire Grauert tube has nonnegative sectional curvature.

The adapted complex structure on $T^R M \setminus M$ can be described as follows. For this purpose let $\theta \in TM \setminus M$ and $x = \pi(\theta)$, where $\pi : TM \to M$ is the projection map. Let γ be a geodesic determined by θ. Choose tangent vectors $v_1, v_2, \cdots, v_{n-1}$ such that $v_1, v_2, \cdots, v_{n-1}, v_n := \frac{\gamma'(0)}{|\gamma'(0)|}$ form an orthonormal basis of $T_x M$.

Denote L_θ the leaf of the foliation $\mathcal{F}$ passing through θ. A vector $\bar{\xi} \in T_\theta TM$ determines a vector field ξ (we call it parallel vector field) along L_θ by defining it to be invariant under two semi-group actions. Namely ξ is invariant under N_τ and the geodesic flow. For this parallel field ξ, we get that $\xi|_{\mathbb{R}}$ is a Jacobi field along γ.

Now choose a set of vectors $\bar{\xi}_1, \bar{\xi}_2, \cdots, \bar{\xi}_n, \bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_n \in T_\theta TM$ satisfying

$$(\pi_*)_\theta(\bar{\xi}_j) = v_j,\ K_\theta(\bar{\xi}_j) = 0$$
$$(\pi_*)_\theta(\bar{\eta}_j) = 0,\ K_\theta(\bar{\eta}_j) = v_j.$$

Let $\xi_1, \xi_2, \cdots, \xi_n, \eta_1, \eta_2, \cdots, \eta_n$ be the parallel vector fields along L_θ determined by $\bar{\xi}_j$ and $\bar{\eta}_j$. There are smooth real valued functions ϕ_{jk} defined on $\mathbb{R} \setminus S_1$ (S_1 is a discrete set) such that

$$\eta_k|_{\mathbb{R}} = \sum_{j=1}^{n} \phi_{jk}\xi_j|_{\mathbb{R}}.$$

The functions ϕ_{jk} have meromorphic extension f_{jk} over the domain

$$D = \{\sigma + i\tau \in \mathbb{C} |\ |\tau| < \frac{R}{\sqrt{g(\theta,\theta)}}\}$$

such that for each j, k, the poles of f_{jk} lies on $\mathbb{R}$ and the matrix $Im(f_{jk})|_{D\setminus\mathbb{R}}$ is invertible.

Let $(e_{jk}) = (Imf_{jk}(i))^{-1}$. Then the complex structure J satisfies

$$J\bar{\xi_h} = \sum_{k=1}^{n} e_{kh} \times [\bar{\eta_k} - \sum_{j=1}^{n} Ref_{jk}(i)\bar{\xi_j}].$$

Remark: Because $\xi_1|\mathbb{R}, \xi_2|\mathbb{R}, \cdots, \xi_{n-1}|\mathbb{R}, \eta_1|\mathbb{R}, \eta_2|\mathbb{R}, \cdots, \eta_{n-1}|\mathbb{R}$ are normal Jacobi fields, while $\xi_n|\mathbb{R}, \eta_n|\mathbb{R}$ are tangential Jacobi fields, for $1 \leq j, k \leq n-1$, we have

$$\phi_{nk} = \phi_{jn} \equiv 0$$

$$f_{nk} = f_{jn} \equiv 0$$

$$e_{nk} = e_{jn} \equiv 0$$

Consider the n-tuples

$$\Xi = (\xi_1, \xi_2, \cdots, \xi_n),\ \ H = (\eta_1, \eta_2, \cdots, \eta_n)$$

and holomorphic n-tuples

$$\Xi^{1,0} = (\xi_1^{1,0}, \xi_2^{1,0}, \cdots, \xi_n^{1,0}),\ \ H^{1,0} = (\eta_1^{1,0}, \eta_2^{1,0}, \cdots, \eta_n^{1,0}),$$

where $\xi_j^{1,0} = \frac{1}{2}(\xi_j - iJ\xi_j)$ and J is the adapted complex structure.

Then we have

$$H(\sigma) = \Xi(\sigma) f(\sigma)$$

$$H^{1,0}(\sigma + i\tau) = \Xi^{1,0}(\sigma + i\tau) f(\sigma + i\tau)$$

$$f(\sigma + i\tau) = (f_{jk}(\sigma + i\tau)),\ \ \sigma \in \mathbb{R} \setminus S_1,\ \ |\tau| < \frac{R}{\sqrt{g(\theta, \theta)}}.$$

Theorem (Lempert and Szöke):

1. The vectors $\xi_1^{1,0}, \xi_2^{1,0}, \cdots \xi_n^{1,0}$ are linearly independent over $\mathbb{C}$ on $D \setminus \mathbb{R}$. The same is true for the vectors $\eta_1^{1,0}, \eta_2^{1,0}, \cdots, \eta_n^{1,0}$.

2. The $2n$ vectors ξ_j, η_k are linearly independent in points $\sigma + i\tau \in D \setminus \mathbb{R}$.

3. The matrix valued meromorphic functions $f(\sigma + i\tau)$ is symmetric (as a matrix) and satisfies

$$f(0) = 0, f'(0) = Id.$$

Moreover, if $\sigma + i\tau \in D, \tau > 0$, then $Imf(\sigma + i\tau)$ is a symmetric, positive definite matrix.

Growth rate of counting functions

Let M be a n-dimensional compact manifold endowed with a Riemannian metric g. For $x \in M$ and each $T > 0$, let

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be the disk of radius T in $T_x M$. Define the counting function $n_T(x, y)$ by

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In other words, $n_T(x, y)$ counts the number of geodesic arcs joining x to y with length $\leq T$.

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$$\int_M n_T(x,y)dy = \int_0^T d\sigma \int_{\mathbb{S}} \sqrt{det(g(J_j(\sigma), J_k(\sigma)))_{j,k=1,2,\cdots,n-1}}\, d\theta,$$

where $J_j, j = 1, 2, \cdots, n-1$ are Jacobi fields along the unique geodesic γ determined by $\theta \in \mathbb{S}$ with initial conditions

$$J_j(0) = 0$$

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Theorem (Gromov): Let M be a n-dimensional compact simply connected manifold endowed with a Riemannian metric g, then

$$\sum_{j=0}^{k-1} dim H_j(\Omega M) \leq \frac{1}{Vol_g(M)} \int_M n_{Ck}(x,y)dy.$$

Now suppose that M has entire Grauert tube. we will derive that $\int_M n_T(x,y)dy$ has polynomial growth and hence M is rationally elliptic.

Crucial observations:

1. Let $f_1 = (f_{jk}),\ j,k = 1,2,\cdots,n-1$. Then there exists a discrete subset $S_2 \subset \mathbb{R}$ such that for $\sigma \in \mathbb{R} \setminus S_2$, we have

$$det(g(J_j(\sigma), J_k(\sigma))_{j,k=1,2,\cdots,n-1} = \frac{1}{det((-f_1^{-1})'(\sigma))}.$$

2. $G(\zeta) := -f^{-1}(\zeta)$ is a matrix valued meromorphic function on $\mathbb{C}$ whose pole lies in a discrete subset of $\mathbb{R}$ and $ImG(\zeta)$ is positive definite for $\zeta = \sigma + i\tau \in \mathbb{C}^+$, where $\mathbb{C}^+$ is the upper half plane.

Theorem (Fatou): Let F be an $n \times n$ matrix valued holomorphic function on the upper half plane $\mathbb{C}^+ = \{\xi \in \mathbb{C} | Im\ \zeta > 0\} \cup (\mathbb{R} \setminus P)$, where P is a discrete subset of $\mathbb{R}$ consisting of poles of F. Suppose that for every $\zeta \in \mathbb{C}^+$, $ImF(\zeta)$ is a symmetric, positive definite matrix, whereas for $\zeta \in \mathbb{R} \setminus P$, $ImF(\zeta) = 0$. Then there is an $n \times n$ symmetric matrix $\mu = (\mu_{jk})$ whose entries are real valued, signed Borel measures on $\mathbb{R}$ such that

1° μ_{jk} does not have mass on any interval which does not contain a pole of F;

2° $\int_{-\infty}^{+\infty} \frac{|d\mu_{jk}(t)|}{1+t^2} < \infty$;

3° μ is positive semidefinite in the sense that for any $(\omega_j) \in \mathbb{R}^n$, the measure $\sum \omega_j \omega_k \mu_{jk}$ is nonnegative;

4° $F'(\zeta) = A + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(\zeta - t)^2},\ \zeta \in \mathbb{C}^+$,

where A is a symmetric, positive semidefinite constant matrix. In fact, we have $A = \lim_{\tau \to +\infty} \frac{ImF(i\tau)}{\tau}$ and $d\mu(\sigma)$ is the weak limit of $ImF(\sigma + i\tau)$ as $\tau \to 0^+$.

Applying the above Fatou's representation theorem to the matrix valued holomorphic function $(-f_1^{-1})$ on the upper half plane, we get

$$(-f_1^{-1})'(\zeta) = A + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(\zeta - t)^2},\ \zeta \in \mathbb{C}^+,$$

As μ does not have mass on any interval which does not contain a pole of $-f_1^{-1}$. This yields that

$$(-f_1^{-1})'(\sigma) = A + \frac{1}{\pi}\sum_j \frac{\mu(t_j)}{(\sigma - t_j)^2},\ \sigma \in \mathbb{R} \setminus \{t_1, t_2, \cdots\},$$

where $\{t_1, t_2, \cdots\}$ are poles of $-f_1^{-1}$.

Key observation: As $f(0) = 0$, we see that 0 is pole of $-f_1^{-1}$. Moreover, we have

$$\mu(0) = \pi Id$$

Then

$$(-f_1^{-1})'(\sigma) = \frac{1}{\sigma^2} Id + B,$$

where $B = A + \frac{1}{\pi} \sum_{t_j \neq 0} \frac{\mu(t_j)}{(\sigma - t_j)^2}$ is positive semidefinite. Then we get

$$\frac{1}{det((-f_1^{-1})'(\sigma))} \leq \sigma^{2n-2}.$$

So

$$\int_M n_T(x,y)dy \leq p(T),$$

where $p(T)$ is a polynomial of degree at most n. It follows that M is rationally elliptic.

approach via rational homotopy theory

Theorem (Grove-Halperin): $M = D(B_1) \cup_E D(B_0)$ is rationally elliptic if and only if E is.

Double disk conjecture (Grove): A compact simply connected manifold with nonnegative sectional curvature is a double disk bundle, i.e, it is obtained as the union of the total spaces of two disk bundles via their common boundary.

Our observation: The total space of an open book decomposition is rationally elliptic if its page is a disk bundle over a rationally elliptic manifold.

An open book decomposition of M^n consists of a codimension-two submanifold N^{n-2}, called the binding, and a fibration $\pi : M \setminus N \to S^1$. The fibres are called the pages.

Question: Can we construct such a special open book decomposition on the sphere bundle of a compact simply connected manifold with nonnegative sectional curvature?

Thank you!