Simplicial volume and isolated, closed totally geodesic submanifolds of codimension one

Yuping Ruan

December 21, 2024

Table of Contents

Table of Contents

Gromov norm

Definition

Let M be a connected, closed, oriented topological manifold. For any singular homology class $\alpha \in H_k(M, \mathbb{R})$, the *Gromov norm* $\|\alpha\|$ of α is defined by

$$\|\alpha\| := \inf \left\{ \sum_{i \in I} |a_i| \,\middle|\, \sum_{i \in I} a_i \sigma_i \text{ is a } k \text{ cycle representing } \alpha \right\}$$

In particular, the Gromov norm of the fundamental class $[M]$ is called the *simplicial volume* of M, which we denote by $\|M\|$.

A rough interpretation of simplicial volume

- Intuitively, the fundamental class of M can be thought of as a triangularization of M.

A rough interpretation of simplicial volume

- Intuitively, the fundamental class of M can be thought of as a triangularization of M.
- For any $p, q \in \mathbb{Z}_+$, we say that M can be triagularized by a total of p/q top dimensional simplices if a q-fold cover of M can be triagularized using a total of p top dimensional simplices.

A rough interpretation of simplicial volume

- Intuitively, the fundamental class of M can be thought of as a triangularization of M.
- For any $p, q \in \mathbb{Z}_+$, we say that M can be triagularized by a total of p/q top dimensional simplices if a q-fold cover of M can be triagularized using a total of p top dimensional simplices.
- $\|M\|$ can be thought of as the

$$\inf \left\{ m \in \mathbb{R}^+ \;\middle|\; \begin{array}{l} M \text{ can be triagularized by a total of} \\ m \text{ top dimensional simplices.} \end{array} \right\}.$$

  In particular, if $\|M\| = 0$, this means that M can be triangularized as efficiently as possible; if $\|M\| > 0$, this means one cannot triangularize M as efficiently as possible.

Example: 2-torus

Fact

When $M = \mathbb{T}^2$, $\|M\| = 0$.

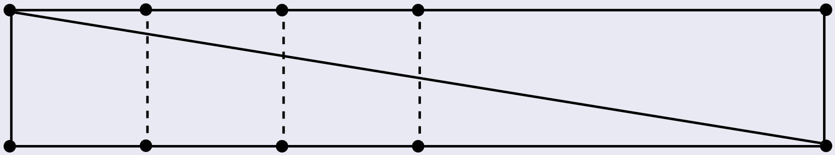
Example: 2-torus

Fact

When $M = \mathbb{T}^2$, $\|M\| = 0$.

Proof.

$$\|M\| \leq \frac{2}{n},\ \forall n \in \mathbb{Z}_+.\ \text{Hence } \|M\| = 0.$$



□

Example: surface of genus $g \geq 2$

Fact

When M is a closed surface with genus $g \geq 2$, we have $\|M\| > 0$.

Example: surface of genus $g \geq 2$

Fact

When M is a closed surface with genus $g \geq 2$, we have $\|M\| > 0$.

Proof. (Gromov, Thurston) Observe the following facts:

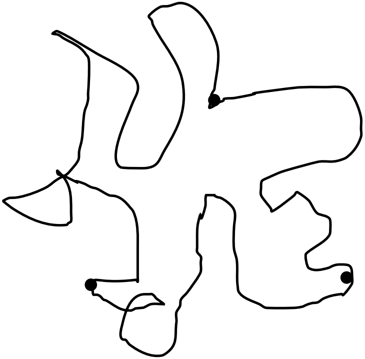
- M admits a Riemannian metric g_0 with constant curvature -1.
- Every triangularization of any finite cover of M can be "deformed" into a triangularization by the same amount of geodesic triangles with respect to the metric g_0.
- Every geodesic triangle has g_0-area at most π.

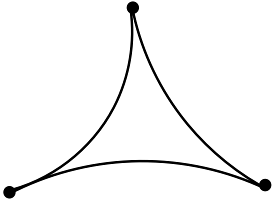
Therefore $\|M\| \geq \mathrm{Area}_{g_0}(M)/\pi > 0$.

Example: surface of genus $g \geq 2$ continued

Fact

When M is a closed surface with genus $g \geq 2$, we have $\|M\| > 0$.

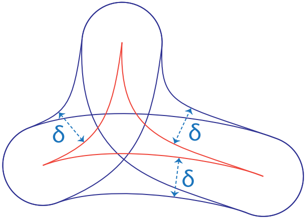




Gromov hyperbolicity

Definition

A geodesic metric space X is δ-*hyperbolic* if every geodesic triangle is δ-thin.
A geodesic metric space X is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.



Picture source: By Stomatapoll - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=22898943.

Example: M with Gromov hyperbolic universal cover

Fact

When the universal cover of M is Gromov hyperbolic, we have $\|M\| > 0$.

Proof. (Mineyev) Observe the following facts:

- The universal cover of M is Gromov hyperbolic.
- Every triangularization of any finite cover of M can be "deformed" into a special triangularization. Moreover, there exists some uniform constant $C > 0$ such that

$$\frac{\#\{\text{simplices in the special triangularization}\}}{\#\{\text{simplices in the original triangularization}\}} \le C$$

- The special triangularization only involves a finite collection of simplices $\{\sigma_1, \ldots, \sigma_m\}$.

Therefore $\|M\| \geq \mathrm{Area}(M)/C \cdot \max_{1\leq k\leq m}\{\mathrm{Area}(\sigma_k)\} > 0$. (Details discussed later.)

Some known results

The simplicial volume $\|M\|$ is positive if

- M is negatively curved. (Gromov 82', Thurston 77')
- M is a locally symmetric space of non-compact type. (Lafont-Schmidt, 06')
- M is nonpositively curved and admits a point with negative curvature (Connell-Wang, 20')
- The universal cover of M is Gromov hyperbolic. (Equivalently, $\pi_1(M)$ is Gromov hyperbolic.) (Mineyev, 01')
- $\pi_1(M)$ is relatively hyperbolic with respect to a collection of fundamental groups of submanifolds (Mineyev-Yaman (preprint), Franceschini, 18')

Conjectures

Conjecture (Gromov)

If M admits a Riemannian metric with nonpositive sectional curvature and negative definite Ricci curvature, then $||M|| > 0$.

Conjecture (Gromov)

If M is aspherical and $||M|| = 0$, then the Euler characteristic of M is zero.

Conjecture (Connell-Wang)

If M admits a Riemannian metric with nonpositive sectional curvature everywhere and negative definite Ricci curvature at some point, then $||M|| > 0$.

Table of Contents

Totally geodesic submanifolds

Definition

A Riemannian submanifold N in a Riemannian manifold M is *totally geodesic* if geodesics in N are also geodesics in M.

Totally geodesic submanifolds

Definition

A Riemannian submanifold N in a Riemannian manifold M is *totally geodesic* if geodesics in N are also geodesics in M.

Examples

- When $M = \mathbb{R}^n$ is equipped with the standard Euclidean metric, lower dimensional subspaces are totally geodesic in M.
- When $M = \mathbb{S}^2$ is equipped with the standard spherical metric, great circles are totally geodesic in M.

Parallel geodesics and Sandwich lemma

Definition (Parallel)

Let X be a simply connected, non-positively curved Riemannian manifold. Let $c_1, c_2 : \mathbb{R} \to X$ be geodesics in X. We say that c_1 and c_2 are *parallel* if there exists some $\delta > 0$ such that the following holds:

- c_1 is contained in a δ-neighborhood of c_2.
- c_2 is contained in a δ-neighborhood of c_1.

Sandwich lemma

If c_1 and c_2 are parallel, they bound a totally geodesic submanifold isometric to $[0, r] \times \mathbb{R}$ in X.

Parallel geodesics and Sandwich lemma

Definition (Parallel)

Let X be a simply connected, non-positively curved Riemannian manifold. Let $c_1, c_2 : \mathbb{R} \to X$ be geodesics in X. We say that c_1 and c_2 are *parallel* if there exists some $\delta > 0$ such that the following holds:

- c_1 is contained in a δ-neighborhood of c_2.
- c_2 is contained in a δ-neighborhood of c_1.

Sandwich lemma

If c_1 and c_2 are parallel, they bound a totally geodesic submanifold isometric to $[0, r] \times \mathbb{R}$ in X.

Some terminologies:

- Flat strip in X: tot, geo. submanifold in X isometric to $[0, r] \times \mathbb{R}$.
- k-flat in X: tot. geo. submanifold in X isometric to $\mathbb{R}^k$. $(k \geq 2)$

Isolatedness

Definition (Isolated)

Let M be a compact, non-positively curved Riemannian manifold with dimension at least two. Let N be a closed, tot. geo. submanifold of codimension-1. We say that N is *isolated* in M if the following holds: Let F be any lift of N in the universal cover X of M. Then

- (No self-intersection) $\gamma F \cap F = \emptyset$ for any $\gamma \in \pi_1(M)$.
- (All geodesic parallel to F is contained in F) If a bi-infinite geodesic in X lies in the r-neighborhood of F for some $r > 0$, then this geodesic is contained in F.

Examples

- M is negatively curved surface, N is the shortest closed geodesic.
- M is analytic and rank-one, a lift of N in the universal cover of M is a codimension-1 flat.

Rank of a geodesic

Definition (Rank of a geodesic)

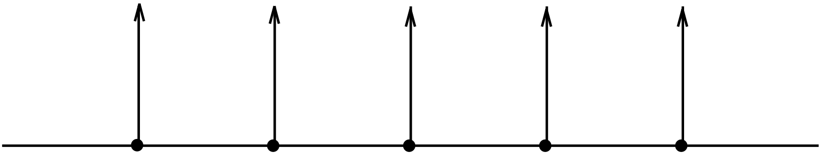
The *rank* of a geodesic is the dimension of the space of parallel Jacobi fields along this geodesic. The rank of a manifold M is the smallest possible rank of geodesics in M.

Rank of a geodesic

Definition (Rank of a geodesic)

The *rank* of a geodesic is the dimension of the space of parallel Jacobi fields along this geodesic. The rank of a manifold M is the smallest possible rank of geodesics in M.

- All geodesics have rank at least 1. (Tangent vector field along a geodesic is a parallel Jacobi field)
- If a geodesic is contained in a k-flat after lifting to the universal cover, then the rank of this geodesic is at least k.



Flats in analytic, nonpositively curved manifolds

Let X be a simply-connected, analytic non-positively curved Riemannian manifold $(\dim \geq 3)$ and F be a codimension-1 flat in X.

Flats in analytic, nonpositively curved manifolds

Let X be a simply-connected, analytic non-positively curved Riemannian manifold ($\dim \geq 3$) and F be a codimension-1 flat in X.
Observe the following:

- If a geodesic is contained in a F, then the rank of this geodesic is at least k.
- The intersection between two different tot. geo. submanifolds is tot. geo.
- Analytic assumption implies that any flat strip in X can be extended to a 2-flat.

Flats in analytic, nonpositively curved manifolds

Let X be a simply-connected, analytic non-positively curved Riemannian manifold ($\dim \geq 3$) and F be a codimension-1 flat in X.
Observe the following:

- If a geodesic is contained in a F, then the rank of this geodesic is at least k.
- The intersection between two different tot. geo. submanifolds is tot. geo.
- Analytic assumption implies that any flat strip in X can be extended to a 2-flat.

Hence if there is a different flat intersecting F, then X has rank at least 2.

Isolatedness

Definition (Isolated)

Let M be a compact, non-positively curved Riemannian manifold with dimension at least two. Let N be a closed, tot. geo. submanifold of codimension-1. We say that N is *isolated* in M if the following holds: Let F be any lift of N in the universal cover X of M. Then

- (No self-intersection) $\gamma F \cap F = \emptyset$ for any $\gamma \in \pi_1(M)$.
- (All geodesic parallel to F is contained in F) If a bi-infinite geodesic in X lies in the r-neighborhood of F for some $r > 0$, then this geodesic is contained in F.

Examples

- M is negatively curved surface, N is the shortest closed geodesic.
- M is analytic and rank-one, a lift of N in the universal cover of M is a codimension-1 flat.

Main theorem

Theorem A (24' Connell-R-Wang)

Let M be a compact, non-positively curved Riemannian manifold with dimension at least two. If it admits an isolated, closed, totally geodesic submanifold of codimension-1, then the simplicial volume $\|M\| > 0$.

Main theorem

Theorem A (24' Connell-R-Wang)

Let M be a compact, non-positively curved Riemannian manifold with dimension at least two. If it admits an isolated, closed, totally geodesic submanifold of codimension-1, then the simplicial volume $\|M\| > 0$.

Corollary (24' Connell-R-Wang)

Let M be a compact, analytic, non-positively curved Riemannian manifold with dimension at least two. If its universal cover admits a codimension one flat. Then exactly one of the following holds:

- $\|M\| > 0$.
- M has non-trivial Euclidean de Rham factors. (In particular, M has rank at least two.

Higher rank manifolds

Higher rank:= "rank$\geq$ 2".

Higher rank manifolds

Higher rank:= "rank$\geq$ 2".

Examples

The following non-positively curved manifolds have higher rank:

- $M_1 \times M_2$, where M_1 and M_2 are non-positively curved.
- $\Gamma \backslash \mathrm{SL}(3, \mathbb{R}) / \mathrm{SO}(3)$, where Γ is a lattice in $\mathrm{SL}(3, \mathbb{R})$.

Higher rank manifolds

Higher rank:= "rank$\geq$ 2".

Examples

The following non-positively curved manifolds have higher rank:

- $M_1 \times M_2$, where M_1 and M_2 are non-positively curved.
- $\Gamma\backslash \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, where Γ is a lattice in $\mathrm{SL}(3,\mathbb{R})$.

Theorem (Rank rigidity theorem, Ballman, Burns-Spatzier, 87')

Let M be a closed Riemannian manifold with non-positive curvature. Then M has higher rank if and only if one of the following holds:

- The universal cover of M splits as a product.
- M is locally symmetric.

Maximal higher rank submanifolds (MHRS)

Let M be a closed, nonpositively curved Riemannian manifold with universal cover X. Let $\mathrm{P} : X \to M$ be the covering map.

- Higher rank submanifolds (HRS) of X:="complete tot. geo. submanifolds of X with rank at least 2."

Maximal higher rank submanifolds (MHRS)

Let M be a closed, nonpositively curved Riemannian manifold with universal cover X. Let $\mathrm{P} : X \to M$ be the covering map.

- Higher rank submanifolds (HRS) of X:= "complete tot. geo. submanifolds of X with rank at least 2."
- Maximal higher rank submanifolds (MHRS) of X:= HRS of X which is not properly contained in another HRS.

Maximal higher rank submanifolds (MHRS)

Let M be a closed, nonpositively curved Riemannian manifold with universal cover X. Let $\mathrm{P} : X \to M$ be the covering map.

- Higher rank submanifolds (HRS) of X:="complete tot. geo. submanifolds of X with rank at least 2."
- Maximal higher rank submanifolds (MHRS) of X:= HRS of X which is not properly contained in another HRS.
- A MHRS in X is closed if its image under P is compact.

Maximal higher rank submanifolds (MHRS)

Let M be a closed, nonpositively curved Riemannian manifold with universal cover X. Let $\mathrm{P} : X \to M$ be the covering map.

- Higher rank submanifolds (HRS) of X:="complete tot. geo. submanifolds of X with rank at least 2."
- Maximal higher rank submanifolds (MHRS) of X:= HRS of X which is not properly contained in another HRS.
- A MHRS in X is closed if its image under P is compact.

Some facts about MHRS:

- Flats are always containd in a MHRS.
- If MHRS (or flats) do not exist, then X is Gromov hyperbolic. In particular, $\|M\| > 0$ is already known due to Mineyev.

Classification of rank-one analytic 4-manifolds with nonpositive curvature

Theorem (Schroeder, 89')

Let M be a closed, rank one, 4-dimensional analytic manifold of nonpositive curvature, and X be the universal cover of M. Then all MHRS of X are closed. Moreover, for any MHRS F, one of the following holds:

(1) F is a 2-flat.

(2) F is a 3-flat.

(3) F is isometric to $\Sigma \times \mathbb{R}$, where Σ is a non-flat 2-dimensional Hadamard manifold. There are two cases:

- (3a) F does not intersect any other MHRS of the same type.
- (3b) F intersects another MHRS of the same type.

Gromov's conjecture restricted to dimension 4

Conjecture (Gromov)

If M is closed, aspherical and $||M|| = 0$, then the Euler characteristic of M is zero.

Difficult!

Gromov's conjecture restricted to dimension 4

Weaker conjecture in dimension 4

In dimension 4, if M is closed, non-positively curved and $||M|| = 0$, then the Euler characteristic of M is zero.

Gromov's conjecture restricted to dimension 4

Weaker conjecture in dimension 4

In dimension 4, if M is closed, non-positively curved and $||M|| = 0$, then the Euler characteristic of M is zero.

Still difficult!

Gromov's conjecture restricted to dimension 4

Weaker conjecture in dimension 4

In dimension 4, if M is closed, analytic, non-positively curved and $||M|| = 0$, then the Euler characteristic of M is zero.

Gromov's conjecture restricted to dimension 4

Weaker conjecture in dimension 4

In dimension 4, if M is closed, analytic, non-positively curved and $||M|| = 0$, then the Euler characteristic of M is zero.

Theorem B (23' Connell-R-Wang)

Let M be a 4-dimensional closed, analytic manifold of non-positive curvature. If the Euler charateristic of M is zero, then M has non-trivial Euclidean de Rham factors. In particular, $\|M\| = 0$.

Gromov's conjecture restricted to dimension 4

Weaker conjecture in dimension 4

In dimension 4, if M is closed, analytic, non-positively curved and $||M|| = 0$, then the Euler characteristic of M is zero.

Theorem B (23' Connell-R-Wang)

Let M be a 4-dimensional closed, analytic manifold of non-positive curvature. If the Euler charateristic of M is zero, then M has non-trivial Euclidean de Rham factors. In particular, $\|M\| = 0$.

Main idea of proof:
If Ricci of M degenerate everywhere, this is known due to Guler-Zheng.

Gromov's conjecture restricted to dimension 4

Weaker conjecture in dimension 4

In dimension 4, if M is closed, analytic, non-positively curved and $||M|| = 0$, then the Euler characteristic of M is zero.

Theorem B (23' Connell-R-Wang)

Let M be a 4-dimensional closed, analytic manifold of non-positive curvature. If the Euler charateristic of M is zero, then M has non-trivial Euclidean de Rham factors. In particular, $\|M\| = 0$.

Main idea of proof:
If Ricci of M degenerate everywhere, this is known due to Guler-Zheng. If M admits points with negative-definite Ricci, we construct a 2-tensor q. Do a first variation of the Gauss-Bonnet-Chern formula along q to show that there are too many 3-flats in the universal cover of M. Hence M must be higher rank. The rest follows from the rank rigidity theorem.

Gromov's conjecture restricted to dimension 4

Conjecture

Let M be a closed, analytic, non-positively curved 4-manifold. Then $||M|| = 0$ if and only if the Euler characteristic of M is zero.

Gromov's conjecture restricted to dimension 4

Conjecture

Let M be a closed, analytic, non-positively curved 4-manifold. Then $||M|| = 0$ if and only if the Euler characteristic of M is zero.

Theorem (Schroeder, 89')

Let F be a MHRS in the universal cover of M. One of the following holds:

(1) F is a 2-flat.

(2) F is a 3-flat.

(3) F is isometric to $\Sigma \times \mathbb{R}$.There are two cases:

- (3a) F does not intersect any other MHRS of the same type.
- (3b) F intersects another MHRS of the same type.

Gromov's conjecture restricted to dimension 4

Conjecture

Let M be a closed, analytic, non-positively curved 4-manifold. Then $||M|| = 0$ if and only if the Euler characteristic of M is zero.

Theorem (Schroeder, 89')

Let F be a MHRS in the universal cover of M. One of the following holds:

(1) F is a 2-flat.

(2) F is a 3-flat.

(3) F is isometric to $\Sigma \times \mathbb{R}$.There are two cases:

- (3a) F does not intersect any other MHRS of the same type.
- (3b) F intersects another MHRS of the same type.

Progress: Conjecture is true when

- Type (3b) does not exist. (Hruska-Kleiner-Hindawi-Mineyev-Yaman)
- Type (2) or (3a) exist. (Theorem A, 24' Connell-R-Wang)

Table of Contents

Straightening argument

Proving $\|M\| > 0$ **via straightening**:

Straightening argument

Proving $\|M\| > 0$ **via straightening**:

- Need a notion of hyperbolicity. (e.g. negative curvature, Gromov hyperbolicity, relative hyperbolicity, etc.)

Straightening argument

Proving $\|M\| > 0$ **via straightening**:

- Need a notion of hyperbolicity. (e.g. negative curvature, Gromov hyperbolicity, relative hyperbolicity, etc.)
- Construct a notion of "volume".

Straightening argument

Proving $\|M\| > 0$ **via straightening**:

- Need a notion of hyperbolicity. (e.g. negative curvature, Gromov hyperbolicity, relative hyperbolicity, etc.)
- Construct a notion of "volume".
- Construct special simplices such that all special simplices have "volume" uniformly bounded by A, where $A > 0$ is a uniform constant.

Straightening argument

Proving $\|M\| > 0$ via straightening:

- Need a notion of hyperbolicity. (e.g. negative curvature, Gromov hyperbolicity, relative hyperbolicity, etc.)
- Construct a notion of "volume".
- Construct special simplices such that all special simplices have "volume" uniformly bounded by A, where $A > 0$ is a uniform constant.
- For any simplex σ define the straightened simplex $\mathrm{st}(\sigma)$ such that $\mathrm{st}(\sigma)$ is a linear combination of special simplices with total weight bounded by C, where $C > 0$ is a uniform constant.

Straightening argument

Proving $\|M\| > 0$ via straightening:

- Need a notion of hyperbolicity. (e.g. negative curvature, Gromov hyperbolicity, relative hyperbolicity, etc.)
- Construct a notion of "volume".
- Construct special simplices such that all special simplices have "volume" uniformly bounded by A, where $A > 0$ is a uniform constant.
- For any simplex σ define the straightened simplex $\mathrm{st}(\sigma)$ such that $\mathrm{st}(\sigma)$ is a linear combination of special simplices with total weight bounded by C, where $C > 0$ is a uniform constant.

Then we have

$$\|M\| \geq \frac{\text{“volume” of } M}{AC} > 0$$

Straightening argument:

- Hyperbolicity.
- "Volume".
- Special simplices with "volume" $\leq A$.
- Straightened simplex $\mathrm{st}(\sigma)$ with $|\mathrm{st}(\sigma)|_{l^1} \leq C$.

Example: geodesic straightening (Gromov, Thurston)

Fact

When M is a closed surface with genus $g \geq 2$, we have $\|M\| > 0$.

Proof. (Gromov, Thurston) Observe the following facts:

- M admits a Riemannian metric g_0 with constant curvature -1.
- Every triangularization of any finite cover of M can be "deformed" into a triangularization by the same amount of geodesic triangles with respect to the metric g_0.
- Every geodesic triangle has g_0-area at most π.

Therefore $\|M\| \geq \mathrm{Area}_{g_0}(M)/\pi > 0$.

Example: geodesic straightening (Gromov, Thurston)

Fact

When M is a closed surface with genus $g \geq 2$, we have $\|M\| > 0$.

Straightening argument:

- Hyperbolicity.
- "Volume".
- Special simplices with "volume" $\leq A$.
- Straightened simplex $\mathrm{st}(\sigma)$ with $|\mathrm{st}(\sigma)|_{l^1} \leq C$.

Example: geodesic straightening (Gromov, Thurston)

Fact

When M is a closed surface with genus $g \geq 2$, we have $\|M\| > 0$.

Straightening argument:

- Hyperbolicity.
- "Volume".
- Special simplices with "volume" $\leq A$.
- Straightened simplex $\mathrm{st}(\sigma)$ with $|\mathrm{st}(\sigma)|_{l^1} \leq C$.

Application:

- Hyperbolicity:=constant negative curvature.
- "Volume":=Riemannian volume.
- Special simplices:=geodesic simplices.
- Straightened simplex $\mathrm{st}(\sigma)$:=geodesic simplex with the same vertices of σ.

Example: barycentric straightening (Lafont-Schmidt, Connell-Wang)

Barycentric construction: Let X, Y be Riemannian manifolds. If there is a smooth function $\Phi : X \times Y \to \mathbb{R}$ such that $\Phi(\cdot; y)$ is strictly convex for any $y \in Y$ which always admits a critical point, then one can construct the corresponding *barycenter map* $\sigma : Y \to X$ by

$$\sigma(y) := \text{the unique critical point of } \Phi(\cdot; y).$$

Example: barycentric straightening (Lafont-Schmidt, Connell-Wang)

Barycentric construction: Let X, Y be Riemannian manifolds. If there is a smooth function $\Phi : X \times Y \to \mathbb{R}$ such that $\Phi(\cdot; y)$ is strictly convex for any $y \in Y$ which always admits a critical point, then one can construct the corresponding *barycenter map* $\sigma : Y \to X$ by

$$\sigma(y) := \text{the unique critical point of } \Phi(\cdot; y).$$

Barycentric simplices: $Y = \Delta$.

Example: barycentric straightening (Lafont-Schmidt, Connell-Wang)

Barycentric construction: Let X, Y be Riemannian manifolds. If there is a smooth function $\Phi : X \times Y \to \mathbb{R}$ such that $\Phi(\cdot; y)$ is strictly convex for any $y \in Y$ which always admits a critical point, then one can construct the corresponding *barycenter map* $\sigma : Y \to X$ by

$$\sigma(y) := \text{the unique critical point of } \Phi(\cdot; y).$$

Barycentric simplices: $Y = \Delta$.

Examples

Let $\sigma : \Delta \to X$ be a singular simplex with vertices $p_0, \dots, p_k$. If X is simply-connected and nonpositively curved, one can define the straightened simplex $\mathrm{st}(\sigma)$ as a barycenter map using the following Φ.

- $\Phi(q; a_0, \dots, a_k) = a_0(d(q, p_0))^2 + \dots a_k(d(q, p_k))^2$. (Theorem A)
- $\Phi(q; a_0, \dots, a_k) = a_0\mathcal{B}(q, p_0) + \dots a_k\mathcal{B}(q, p_k)$, where $\mathcal{B}$ is a weighted average of Busemann functions. (Besson-Courtois-Gallot, LS, CW)

Example: barycentric straightening (Lafont-Schmidt)

Theorem (Lafont-Schmidt)

When $M = \Gamma\backslash \mathrm{SL}(4,n)/\mathrm{SO}(4)$ with Γ a co-compact lattice of $\mathrm{SL}(4,n)$, we have $\|M\| > 0$.

Straightening argument:

- Hyperbolicity.
- "Volume".
- Special simplices with "volume" $\leq A$.
- Straightened simplex $\mathrm{st}(\sigma)$ with $|\mathrm{st}(\sigma)|_{l^1} \leq C$.

Example: barycentric straightening (Lafont-Schmidt)

Theorem (Lafont-Schmidt)

When $M = \Gamma\backslash \mathrm{SL}(4,n)/\mathrm{SO}(4)$ with Γ a co-compact lattice of $\mathrm{SL}(4,n)$, we have $\|M\| > 0$.

Straightening argument:

- Hyperbolicity.
- "Volume".
- Special simplices with "volume" $\leq A$.
- Straightened simplex $\mathrm{st}(\sigma)$ with $|\mathrm{st}(\sigma)|_{l^1} \leq C$.

Application:

- Hyperbolicity:="Lots of negative curvature, not very many zero curvature" (Eigenvalue matching, Connell-Farb, 03').
- "Volume":=Riemannian volume.
- Special simplices:=barycentric simplices (Besson-Courtois-Gallot, 90's).
- Straightened simplex $\mathrm{st}(\sigma)$:=barycentric simplex with the same vertices of σ.

Example: barycentric straightening (Connell-Wang)

Theorem (Connell-Wang)

When M is a closed, nonpositively curved Riemannian manifold with strictly negative curvature near a point p_0, we have $\|M\| > 0$.

Straightening argument:

- Hyperbolicity.
- "Volume".
- Special simplices with "volume" $\leq A$.
- Straightened simplex $\mathrm{st}(\sigma)$ with $|\mathrm{st}(\sigma)|_{l^1} \leq C$.

Example: barycentric straightening (Connell-Wang)

Theorem (Connell-Wang)

When M is a closed, nonpositively curved Riemannian manifold with strictly negative curvature near a point p_0, we have $\|M\| > 0$.

Straightening argument:

- Hyperbolicity.
- "Volume".
- Special simplices with "volume" $\leq A$.
- Straightened simplex $\mathrm{st}(\sigma)$ with $|\mathrm{st}(\sigma)|_{l^1} \leq C$.

Application:

- Hyperbolicity:=negative curvature near a point.
- "Volume" $:= \rho(x)d\mathrm{vol}$, where $\rho(x)$ is supported near p_0. (Local volume.)
- Special simplices:=barycentric simplices (Besson-Courtois-Gallot).
- Straightened simplex $\mathrm{st}(\sigma)$:=barycentric simplex with the same vertices of σ.

Example: Gromov hyperbolicity (Mineyev)

Fact

When the universal cover of M is Gromov hyperbolic, we have $\|M\| > 0$.

Proof. (Mineyev) Observe the following facts:

- The universal cover of M is Gromov hyperbolic.
- Every triangularization of any finite cover of M can be "deformed" into a special triangularization. Moreover, there exists some uniform constant $C > 0$ such that

$$\frac{\#\{\text{simplices in the special triangularization}\}}{\#\{\text{simplices in the original triangularization}\}} \leq C$$

- The special triangularization only involves a finite collection of simplices $\{\sigma_1, \ldots, \sigma_m\}$.

Therefore $\|M\| \geq \mathrm{Area}(M)/C \cdot \max_{1\leq k\leq m}\{\mathrm{Area}(\sigma_k)\} > 0$. (Details discussed later.)

Example: Gromov hyperbolicity (Mineyev)

Fact

When the universal cover of M is Gromov hyperbolic, we have $\|M\| > 0$.

Straightening argument:

- Hyperbolicity.
- "Volume".
- Special simplices with "volume" $\leq A$.
- Straightened simplex $\mathrm{st}(\sigma)$ with $|\mathrm{st}(\sigma)|_{l^1} \leq C$.

Example: Gromov hyperbolicity (Mineyev)

Fact

When the universal cover of M is Gromov hyperbolic, we have $\|M\| > 0$.

Straightening argument:

- Hyperbolicity.
- "Volume".
- Special simplices with "volume" $\leq A$.
- Straightened simplex $\mathrm{st}(\sigma)$ with $|\mathrm{st}(\sigma)|_{l^1} \leq C$.

Application:

- Hyperbolicity:=Gromov hyperbolicity of the universal cover.
- "Volume":=Riemannian volume.
- Special simplices:=A finite collection of simplices $\{\sigma_1, \ldots, \sigma_m\}$.
- Straightened simplex $\mathrm{st}(\sigma) \in \mathrm{span}\{\sigma_1, \ldots, \sigma_m\}$ with $|\mathrm{st}(\sigma)|_{l^1} < C$.

Example: Gromov hyperbolicity (Mineyev)

Fact

When the universal cover of M is Gromov hyperbolic, we have $\|M\| > 0$.

Straightening argument:

- Hyperbolicity.
- "Volume".
- Special simplices with "volume" $\leq A$.
- Straightened simplex $\mathrm{st}(\sigma)$ with $|\mathrm{st}(\sigma)|_{l^1} \leq C$.

Application:

- Hyperbolicity:=Gromov hyperbolicity of the universal cover.
- "Volume":=Riemannian volume.
- Special simplices:=A finite collection of simplices $\{\sigma_1, \ldots, \sigma_m\}$.
- Straightened simplex $\mathrm{st}(\sigma) \in \mathrm{span}\{\sigma_1, \ldots, \sigma_m\}$ with $|\mathrm{st}(\sigma)|_{l^1} < C$.

Question: What are the constructions of special/straightened simplices?

Mineyev's approach

Notations: M is a compact Riemannian manifold. X is the universal cover of M which is Gromov hyperbolic. $\Gamma := \pi_1(M)$. $C_k(X; \mathbb{R})$ the $\mathbb{R}$-vector space of all k-dimensional singular chains in X.

Mineyev's approach

Notations: M is a compact Riemannian manifold. X is the universal cover of M which is Gromov hyperbolic. $\Gamma := \pi_1(M)$. $C_k(X;\mathbb{R})$ the $\mathbb{R}$-vector space of all k-dimensional singular chains in X.

Proposition ("fake version" of Mineyev, 01)

There is a $\mathbb{R}[\Gamma]$-chain homomorphism $h_\bullet : C_\bullet(X;\mathbb{R}) \to C_\bullet(X;\mathbb{R})$ such that the following holds:

- $h_k = \mathrm{Id}$ whenever $k \leq 0$.
- When $k \geq 2$, for any k-simplex σ, $|h_k(\sigma)|_{l^1} \leq C(k)$ for some uniform constant $C(k)$ which only depends on k. (Bounded at level k.)
- When $k \geq 2$, the image of h_k is contained in a finitely generated $\mathbb{R}[\Gamma]$-module.

Remark: This is not the original version of Mineyev's result. The original version uses cellular homology on $E\Gamma$ instead of singular homology on X. (See the next slide.)

Mineyev's approach

Construction of $E\Gamma$: Start with points in Γ.

Mineyev's approach

Construction of $E\Gamma$: Start with points in Γ. Glue 1-simplices to all ordered pair of points (γ_0, γ_1) in Γ.

Mineyev's approach

Construction of $E\Gamma$: Start with points in Γ. Glue 1-simplices to all ordered pair of points (γ_0, γ_1) in Γ. Glue 2-simplices to all ordered triple of points $(\gamma_0, \gamma_1, \gamma_2)$ in Γ......

Mineyev's approach

Construction of $E\Gamma$: Start with points in Γ. Glue 1-simplices to all ordered pair of points (γ_0, γ_1) in Γ. Glue 2-simplices to all ordered triple of points $(\gamma_0, \gamma_1, \gamma_2)$ in Γ......
Cellular chains of $E\Gamma$: $C_k(E\Gamma; \mathbb{R})$:=$\mathbb{R}$-vector space spanned by all k-simplices in $E\Gamma$ (equivalently, $(k+1)$-ordered tuples of Γ).

Mineyev's approach

Construction of $E\Gamma$: Start with points in Γ. Glue 1-simplices to all ordered pair of points (γ_0, γ_1) in Γ. Glue 2-simplices to all ordered triple of points $(\gamma_0, \gamma_1, \gamma_2)$ in Γ......

Cellular chains of $E\Gamma$: $C_k(E\Gamma;\mathbb{R})$:=$\mathbb{R}$-vector space spanned by all k-simplices in $E\Gamma$ (equivalently, $(k+1)$-ordered tuples of Γ).

Proposition (Mineyev, 01)

There is a $\mathbb{R}[\Gamma]$-chain homomorphism $h_\bullet : C_\bullet(E\Gamma;\mathbb{R}) \to C_\bullet(E\Gamma;\mathbb{R})$ such that the following holds:

- $h_k = \mathrm{Id}$ whenever $k \leq 0$.
- When $k \geq 2$, for any k-simplex σ, $|h_k(\sigma)|_{l^1} \leq C(k)$ for some uniform constant $C(k)$ which only depends on k. (Bounded at level k.)
- When $k \geq 2$, the image of h_k is contained in a finitely generated $\mathbb{R}[\Gamma]$-module.

Rips complex

Word metric on Γ: Let S be a finite generating set of Γ closed under inverse. The corresponding word metric is defined by

$$d_S(\gamma_1, \gamma_2) := \min\{L \in \mathbb{Z}_{\geq 0} : \gamma_1 \gamma_2^{-1} = \gamma_{(1)} \cdots \gamma_{(L)}\}.$$

Rips complex

Word metric on Γ: Let S be a finite generating set of Γ closed under inverse. The corresponding word metric is defined by

$$d_S(\gamma_1, \gamma_2) := \min\{L \in \mathbb{Z}_{\geq 0} : \gamma_1 \gamma_2^{-1} = \gamma_{(1)} \cdots \gamma_{(L)}\}.$$

Construction of Rips complex $Y = Y_{\Gamma,K}$: Fix a large number $K > 0$ and a generating set S of Γ.

Rips complex

Word metric on Γ: Let S be a finite generating set of Γ closed under inverse. The corresponding word metric is defined by

$$d_S(\gamma_1, \gamma_2) := \min\{L \in \mathbb{Z}_{\geq 0} : \gamma_1\gamma_2^{-1} = \gamma_{(1)} \cdots \gamma_{(L)}\}.$$

Construction of Rips complex $Y = Y_{\Gamma,K}$: Fix a large number $K > 0$ and a generating set S of Γ. Start with points in Γ.

Rips complex

Word metric on Γ: Let S be a finite generating set of Γ closed under inverse. The corresponding word metric is defined by

$$d_S(\gamma_1, \gamma_2) := \min\{L \in \mathbb{Z}_{\geq 0} : \gamma_1 \gamma_2^{-1} = \gamma_{(1)} \cdots \gamma_{(L)}\}.$$

Construction of Rips complex $Y = Y_{\Gamma, K}$: Fix a large number $K > 0$ and a generating set S of Γ. Start with points in Γ. Glue 1-simplices to all pair of points (γ_0, γ_1) in Γ if $d_S(\gamma_0, \gamma_1) \leq K$.

Rips complex

Word metric on Γ: Let S be a finite generating set of Γ closed under inverse. The corresponding word metric is defined by

$$d_S(\gamma_1, \gamma_2) := \min\{L \in \mathbb{Z}_{\geq 0} : \gamma_1 \gamma_2^{-1} = \gamma_{(1)} \cdots \gamma_{(L)}\}.$$

Construction of Rips complex $Y = Y_{\Gamma,K}$: Fix a large number $K > 0$ and a generating set S of Γ. Start with points in Γ. Glue 1-simplices to all pair of points (γ_0, γ_1) in Γ if $d_S(\gamma_0, \gamma_1) \leq K$. Glue 2-simplices to all triple of points $(\gamma_0, \gamma_1, \gamma_2)$ in Γ if $d_S(\gamma_i, \gamma_j) \leq K$ for any $i, j \in \{0, 1, 2\}$......

Rips complex

Word metric on Γ: Let S be a finite generating set of Γ closed under inverse. The corresponding word metric is defined by

$$d_S(\gamma_1, \gamma_2) := \min\{L \in \mathbb{Z}_{\geq 0} : \gamma_1 \gamma_2^{-1} = \gamma_{(1)} \cdots \gamma_{(L)}\}.$$

Construction of Rips complex $Y = Y_{\Gamma,K}$: Fix a large number $K > 0$ and a generating set S of Γ. Start with points in Γ. Glue 1-simplices to all pair of points (γ_0, γ_1) in Γ if $d_S(\gamma_0, \gamma_1) \leq K$. Glue 2-simplices to all triple of points $(\gamma_0, \gamma_1, \gamma_2)$ in Γ if $d_S(\gamma_i, \gamma_j) \leq K$ for any $i, j \in \{0, 1, 2\}$......
Properties:

- X is Gromov hyperbolic if and only if (Γ, d_S) is Gromov hyperbolic. (They are quasi-isometric due to Milnor-Svarc)

Rips complex

Word metric on Γ: Let S be a finite generating set of Γ closed under inverse. The corresponding word metric is defined by

$$d_S(\gamma_1, \gamma_2) := \min\{L \in \mathbb{Z}_{\geq 0} : \gamma_1 \gamma_2^{-1} = \gamma_{(1)} \cdots \gamma_{(L)}\}.$$

Construction of Rips complex $Y = Y_{\Gamma,K}$: Fix a large number $K > 0$ and a generating set S of Γ. Start with points in Γ. Glue 1-simplices to all pair of points (γ_0, γ_1) in Γ if $d_S(\gamma_0, \gamma_1) \leq K$. Glue 2-simplices to all triple of points $(\gamma_0, \gamma_1, \gamma_2)$ in Γ if $d_S(\gamma_i, \gamma_j) \leq K$ for any $i, j \in \{0, 1, 2\}$......
Properties:

- X is Gromov hyperbolic if and only if (Γ, d_S) is Gromov hyperbolic. (They are quasi-isometric due to Milnor-Svarc)
- Y is finite dimensional.

Rips complex

Word metric on Γ: Let S be a finite generating set of Γ closed under inverse. The corresponding word metric is defined by

$$d_S(\gamma_1, \gamma_2) := \min\{L \in \mathbb{Z}_{\geq 0} : \gamma_1 \gamma_2^{-1} = \gamma_{(1)} \cdots \gamma_{(L)}\}.$$

Construction of Rips complex $Y = Y_{\Gamma, K}$: Fix a large number $K > 0$ and a generating set S of Γ. Start with points in Γ. Glue 1-simplices to all pair of points (γ_0, γ_1) in Γ if $d_S(\gamma_0, \gamma_1) \leq K$. Glue 2-simplices to all triple of points $(\gamma_0, \gamma_1, \gamma_2)$ in Γ if $d_S(\gamma_i, \gamma_j) \leq K$ for any $i, j \in \{0, 1, 2\}$......

Properties:

- X is Gromov hyperbolic if and only if (Γ, d_S) is Gromov hyperbolic. (They are quasi-isometric due to Milnor-Svarc)
- Y is finite dimensional.
- There are only finitely many k-simplices in Y up to Γ action.

Rips complex

Word metric on Γ: Let S be a finite generating set of Γ closed under inverse. The corresponding word metric is defined by

$$d_S(\gamma_1, \gamma_2) := \min\{L \in \mathbb{Z}_{\geq 0} : \gamma_1 \gamma_2^{-1} = \gamma_{(1)} \cdots \gamma_{(L)}\}.$$

Construction of Rips complex $Y = Y_{\Gamma,K}$: Fix a large number $K > 0$ and a generating set S of Γ. Start with points in Γ. Glue 1-simplices to all pair of points (γ_0, γ_1) in Γ if $d_S(\gamma_0, \gamma_1) \leq K$. Glue 2-simplices to all triple of points $(\gamma_0, \gamma_1, \gamma_2)$ in Γ if $d_S(\gamma_i, \gamma_j) \leq K$ for any $i, j \in \{0, 1, 2\}$......
Properties:

- X is Gromov hyperbolic if and only if (Γ, d_S) is Gromov hyperbolic. (They are quasi-isometric due to Milnor-Svarc)
- Y is finite dimensional.
- There are only finitely many k-simplices in Y up to Γ action.
- When Γ is Gromov hyperbolic and $K \gg 1$, Y is contractible.

Mineyev's approach

Proposition (Mineyev, 01)

There is a $\mathbb{R}[\Gamma]$-chain homomorphism $h_\bullet : C_\bullet(E\Gamma;\mathbb{R}) \to C_\bullet(E\Gamma;\mathbb{R})$ such that the following holds:

- $h_k = \mathrm{Id}$ whenever $k \leq 0$.
- When $k \geq 2$, for any k-simplex σ, $|h_k(\sigma)|_{l^1} \leq C(k)$ for some uniform constant $C(k)$ which only depends on k. (Bounded at level k.)
- When $k \geq 2$, the image of h_k is contained in a finitely generated $\mathbb{R}[\Gamma]$-module.

Mineyev's approach

Proposition (Mineyev, 01)

There is a $\mathbb{R}[\Gamma]$-chain homomorphism $h_\bullet : C_\bullet(E\Gamma;\mathbb{R}) \to C_\bullet(E\Gamma;\mathbb{R})$ such that the following holds:

- $h_k = \mathrm{Id}$ whenever $k \leq 0$.
- When $k \geq 2$, for any k-simplex σ, $|h_k(\sigma)|_{l^1} \leq C(k)$ for some uniform constant $C(k)$ which only depends on k. (Bounded at level k.)
- When $k \geq 2$, the image of h_k is contained in a finitely generated $\mathbb{R}[\Gamma]$-module.

Idea of proof: Let Y be a contractible Rips complex of Γ. Then one can construct the following $\mathbb{R}[\Gamma]$-chain homomorphisms

$$\phi_\bullet : C_\bullet(E\Gamma;\mathbb{R}) \to C_\bullet(Y;\mathbb{R}) \text{ and } \psi_\bullet : C_\bullet(Y;\mathbb{R}) \to C_\bullet(E\Gamma;\mathbb{R})$$

which are bounded at level ≥ 2. ($h_\bullet = \psi_\bullet \circ \phi_\bullet$.)

Mineyev's approach

Boundary map: Let $\sigma = (p_0, \ldots, p_k)$ be a k-simplex. We define the boundary operator by

$$\partial_k(\sigma) := \sum_{j=0}^{k} (-1)^j (p_0, \ldots, \widehat{p_j}, \ldots, p_k).$$

Mineyev's approach

Boundary map: Let $\sigma = (p_0, \ldots, p_k)$ be a k-simplex. We define the boundary operator by

$$\partial_k(\sigma) := \sum_{j=0}^{k} (-1)^j (p_0, \ldots, \widehat{p_j}, \ldots, p_k).$$

Properties of the $\mathbb{R}[\Gamma]$-chain homomorphism:
$\phi_\bullet : C_\bullet(E\Gamma; \mathbb{R}) \to C_\bullet(Y; \mathbb{R})$

- $\partial_k^Y \circ \phi_k = \phi_{k-1} \circ \partial_k^{E\Gamma}$.
- For any $k \geq 2$ and any k-simplex σ in $E\Gamma$, there exists some uniform constant $C(k)$ such that $|\phi_k(\sigma)|_{l^1} \leq C(k)$.

Mineyev's approach

Boundary map: Let $\sigma = (p_0, \dots, p_k)$ be a k-simplex. We define the boundary operator by

$$\partial_k(\sigma) := \sum_{j=0}^{k} (-1)^j (p_0, \dots, \widehat{p_j}, \dots, p_k).$$

Properties of the $\mathbb{R}[\Gamma]$-chain homomorphism:
$\phi_\bullet : C_\bullet(E\Gamma; \mathbb{R}) \to C_\bullet(Y; \mathbb{R})$

- $\partial_k^Y \circ \phi_k = \phi_{k-1} \circ \partial_k^{E\Gamma}$.
- For any $k \geq 2$ and any k-simplex σ in $E\Gamma$, there exists some uniform constant $C(k)$ such that $|\phi_k(\sigma)|_{l^1} \leq C(k)$. In particular, $|\partial_k^Y \circ \phi_k(\sigma)|_{l^1} \leq (k+1)C(k)$.

Boundary map: Let $\sigma = (p_0, \ldots, p_k)$ be a k-simplex. We define the boundary operator by

$$\partial_k(\sigma) := \sum_{j=0}^{k} (-1)^j (p_0, \ldots, \widehat{p_j}, \ldots, p_k).$$

Properties of the $\mathbb{R}[\Gamma]$-chain homomorphism:
$\phi_\bullet : C_\bullet(E\Gamma; \mathbb{R}) \to C_\bullet(Y; \mathbb{R})$

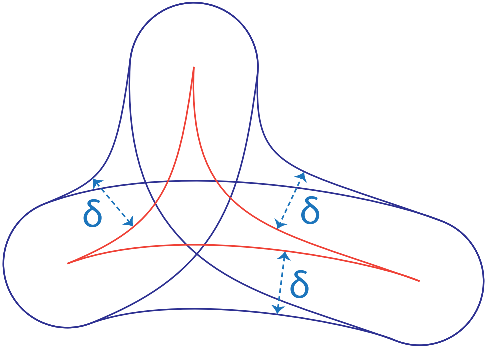
- $\partial_k^Y \circ \phi_k = \phi_{k-1} \circ \partial_k^{E\Gamma}$.
- For any $k \geq 2$ and any k-simplex σ in $E\Gamma$, there exists some uniform constant $C(k)$ such that $|\phi_k(\sigma)|_{l^1} \leq C(k)$. In particular, $|\partial_k^Y \circ \phi_k(\sigma)|_{l^1} \leq (k+1)C(k)$.

Focus on $k = 2$: Let $\beta(\gamma_0, \gamma_1) := \phi_1(\gamma_0, \gamma_1)$, then

$$|\beta(\gamma_0, \gamma_1) + \beta(\gamma_1, \gamma_2) + \beta(\gamma_2, \gamma_0)| \leq 3C(1).$$

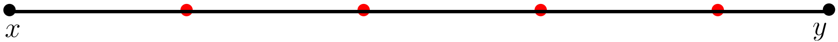
Bicombing construction

Main idea: Start with geodesic triangles. Replace each edge by a convex combination of nearby oriented paths. δ-hyperbolicity implies that these new oriented paths have a lot of cancellations.



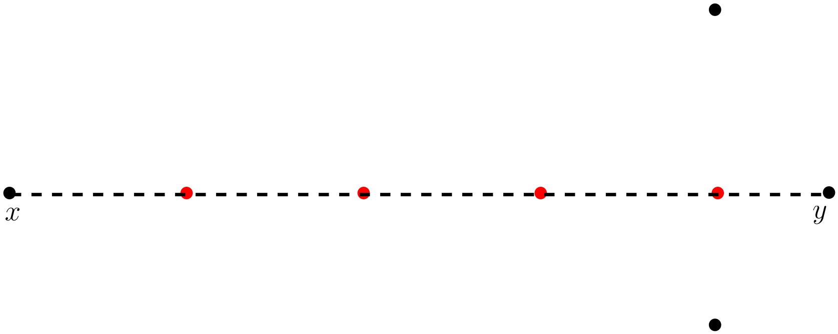
Bicombing construction

Step 1: Cut long geodesics $[x, y]$.



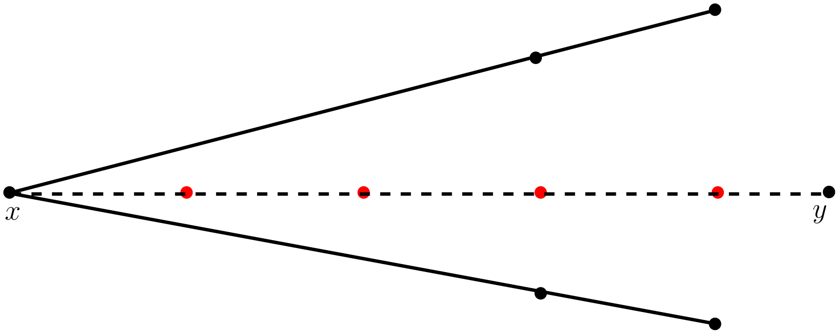
Bicombing construction

Step 2: Construction of $f(x, y)$.



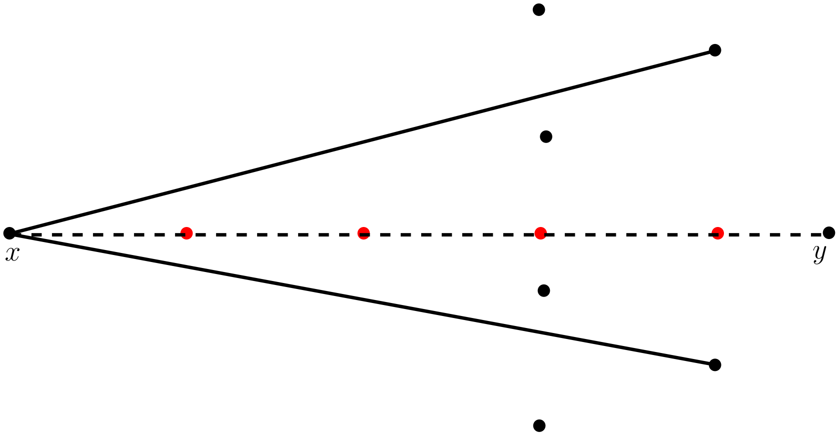
Bicombing construction

Step 2: Construction of $f(x, y)$.



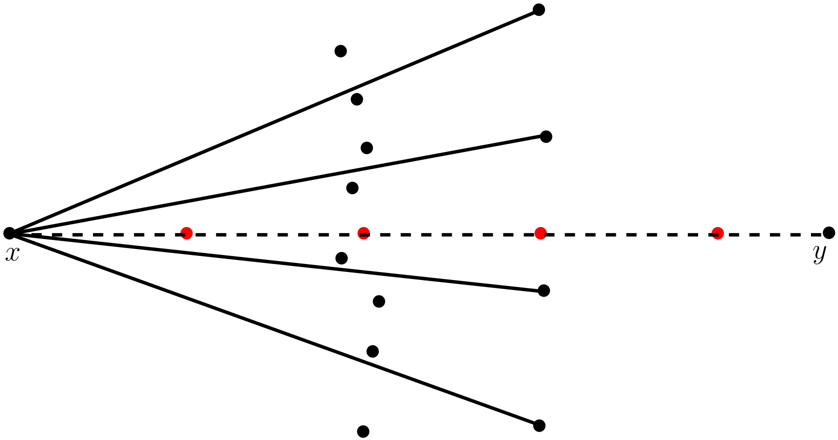
Bicombing construction

Step 2: Construction of $f(x, y)$.



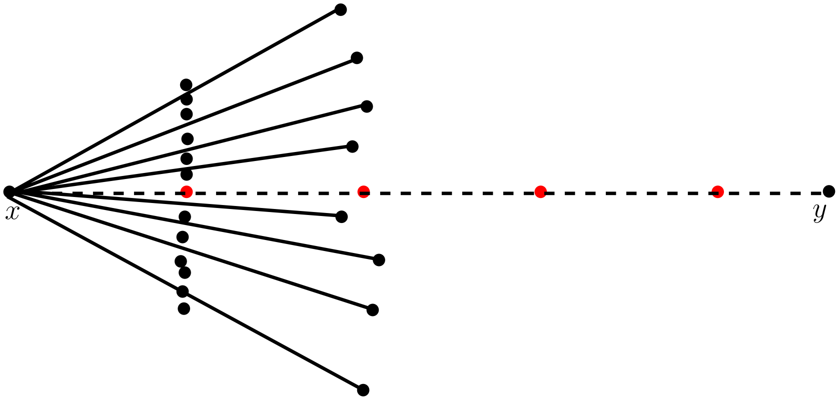
Bicombing construction

Step 2: Construction of $f(x, y)$.



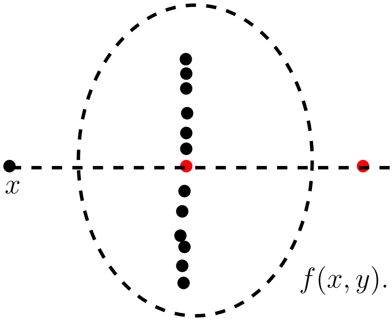
Bicombing construction

Step 2: Construction of $f(x, y)$.



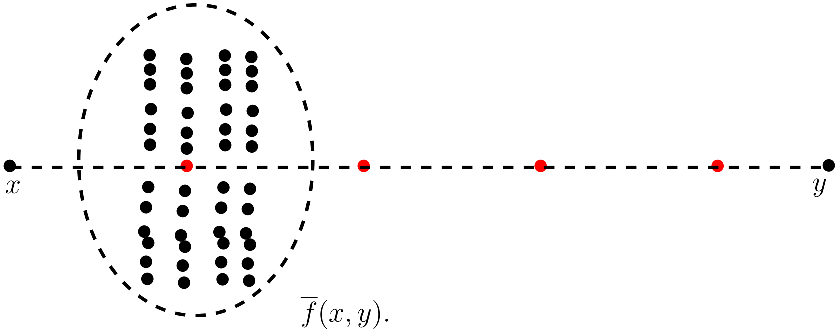
Bicombing construction

Step 2: Construction of $f(x, y)$.

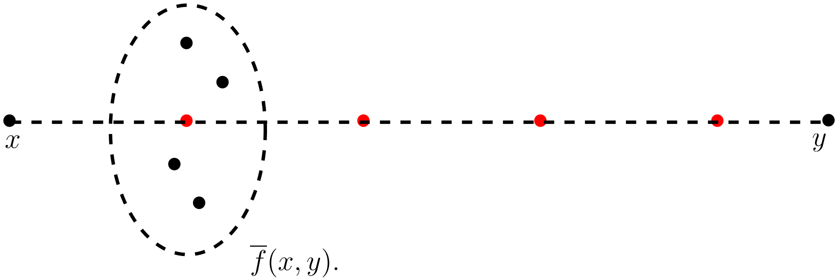


Bicombing construction

Step 2.5: Construction of $\overline{f}(x, y)$.

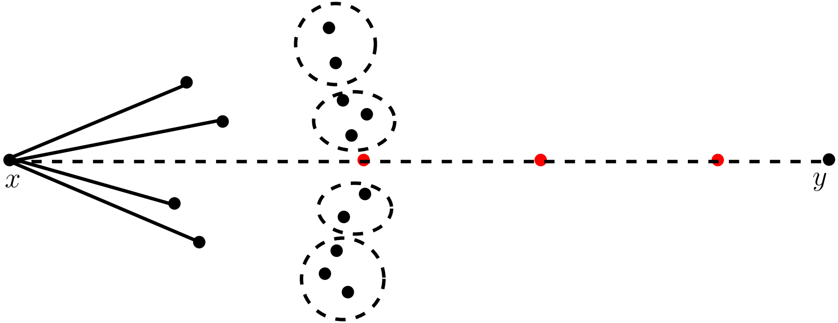


Step 3: Use $f(x, y)$ to construct desired nearby paths.



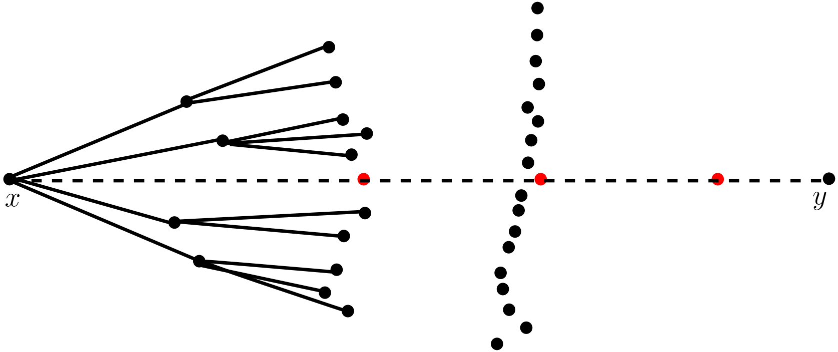
Bicombing construction

Step 3: Use flowers to construct desired nearby paths.



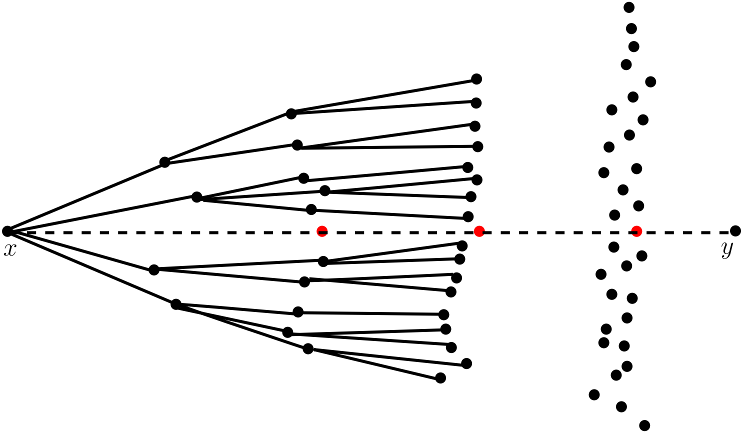
Bicombing construction

Step 3: Use flowers to construct desired nearby paths.



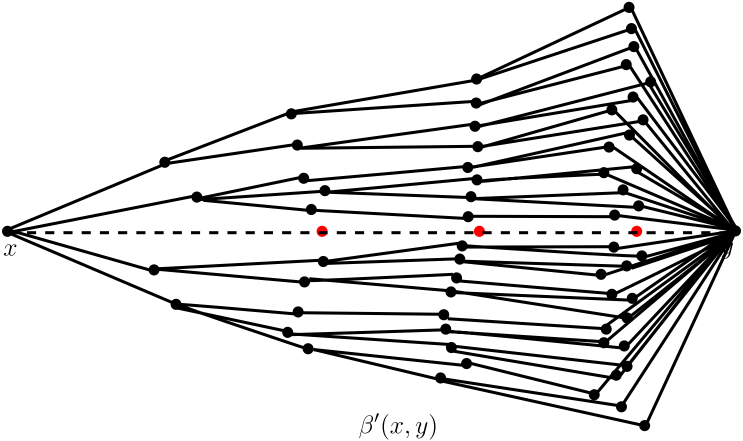
Bicombing construction

Step 3: Use flowers to construct desired nearby paths.



Bicombing construction

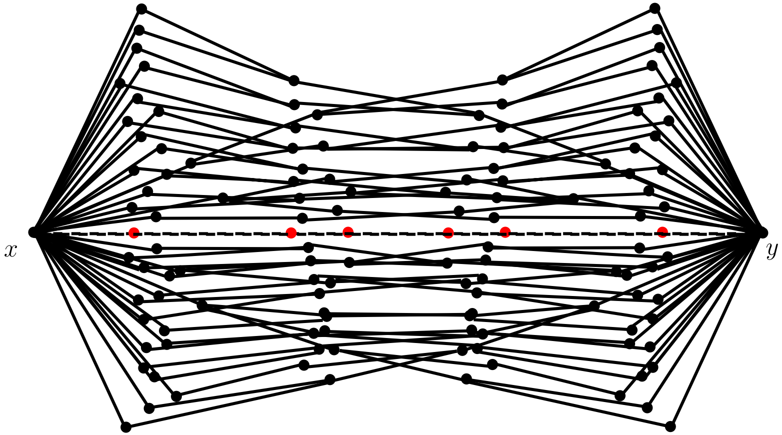
Step 3: Use flowers to construct desired nearby paths.



$\beta'(x, y)$
Arrows go from x to y.

Bicombing construction

Step 4: Anti-symmetrize.



$\beta(x, y)$.

Arrows go from x to y.

Bicombing construction

Step 4: Anti-symmetrize.

$$\beta(x,y) = \frac{1}{2}\left(\beta'(x,y) - \beta'(y,x)\right)$$

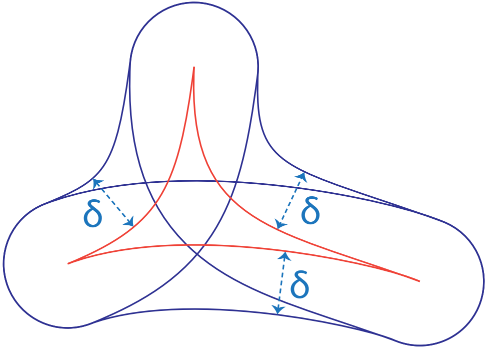


Table of Contents

Review: Straightening argument

Proving $\|M\| > 0$ **via straightening**:

- Need a notion of hyperbolicity. (e.g. negative curvature, Gromov hyperbolicity, relative hyperbolicity, etc.)
- Construct a notion of "volume".
- Construct special simplices such that all special simplices have "volume" uniformly bounded by A, where $A > 0$ is a uniform constant.
- For any simplex σ define the straightened simplex $\mathrm{st}(\sigma)$ such that $\mathrm{st}(\sigma)$ is a linear combination of special simplices with total weight bounded by C, where $C > 0$ is a uniform constant.

Then we have

$$\|M\| \geq \frac{\text{“volume” of } M}{AC} > 0$$

Review: Main theorem

Definition (Isolated)

Let M be a compact, non-positively curved Riemannian manifold with dimension at least two. Let N be a closed, tot. geo. submanifold of codimension-1. We say that N is *isolated* in M if the following holds: Let F be any lift of N in the universal cover X of M. Then

- (No self-intersection) $\gamma F \cap F = \emptyset$ for any $\gamma \in \pi_1(M)$.
- (All geodesic parallel to F is contained in F) If a bi-infinite geodesic in X lies in the r-neighborhood of F for some $r > 0$, then this geodesic is contained in F.

Theorem A (24' Connell-R-Wang)

Let M be a compact, non-positively curved Riemannian manifold with dimension at least two. If it admits an isolated, closed, totally geodesic submanifold of codimension-1, then the simplicial volume $\|M\| > 0$.

Notations

Notations for the rest of this talk:

- M: Compact non-positively curved Riemannian manifold.
- N: An isolated, closed, tot. geo. submanifold of M.
- X: Universal cover of M.
- F: A lift of N. (For simplicity, we call lifts of N as "flats".)
- Γ: Fundamental group of M.
- $[x, y]$: Geodesic segment connecting x, y in X.
- $\text{Proj}_F : X \to F$: orthogonal projection onto F

Isolated condition

- (No self-intersection) $\gamma F \cap F = \emptyset$ for any $\gamma \in \Gamma$.
- (All geodesic parallel to F is contained in F) If a bi-infinite geodesic in X lies in the r-neighborhood of F for some $r > 0$, then this geodesic is contained in F.

Isolated condition

- (No self-intersection) $\gamma F \cap F = \emptyset$ for any $\gamma \in \Gamma$.
- (All geodesic parallel to F is contained in F) If a bi-infinite geodesic in X lies in the r-neighborhood of F for some $r > 0$, then this geodesic is contained in F.

Hyperbolicity

Isolated condition

- (No self-intersection) $\gamma F \cap F = \emptyset$ for any $\gamma \in \Gamma$.
- (All geodesic parallel to F is contained in F) If a bi-infinite geodesic in X lies in the r-neighborhood of F for some $r > 0$, then this geodesic is contained in F.

Key lemma-1: Hyperbolicity perpendicular to the "flats"

There exists some $\delta > 0$ such that for any distinct $x, y, z \in X$ satisfying $y, z \in F$ and $[x, y] \perp F$, the geodesic triangle with vertices x, y, z is δ-thin.

Compare to

Definition (Gromov hyperbolicity)

δ-*hyperbolicity*: Every geodesic triangle is δ-thin.

Key lemma-1 implies the following Key lemma:

Key lemma-2: observing $d([x,y],F)$ via $d(\mathrm{Proj}_F(x),\mathrm{Proj}_F(y))$

Let $q_1,q_2 \in X \setminus F$. Denoted by $r_j = \mathrm{Proj}_F(q_j)$, j=1,2.

(1) For any $\epsilon > 0$, there exist some $R_1(\epsilon) > 0$ such that if $d(r_1,r_2) \geq R_1(\epsilon)$, then $d([q_1,q_2],F) \leq \epsilon$. In other words, if $d([q_1,q_2],F) > \epsilon$, then $d(r_1,r_2) < R_1(\epsilon)$.

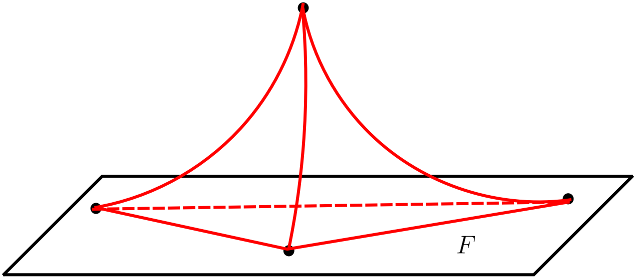
(2) If we assume in addition that q_1,q_2 are on the same side of F, for any $r>0$, $R>0$, there exists some $c_1(r,R)>0$ such that if $d(q_j,F) \geq r$, $j=1,2$ and $d(r_1,r_2) \leq R$, then $d([q_1,q_2],F) \geq c_1(r,R)$.

Vaguely speaking, the claim "$d([x,y],F)$ **is small if and only if** $d(\mathrm{Proj}_F(x),\mathrm{Proj}_F(y))$ **is large**" is true unless

- $[x,y]$ intersects F.
- one of x,y is too close to F.

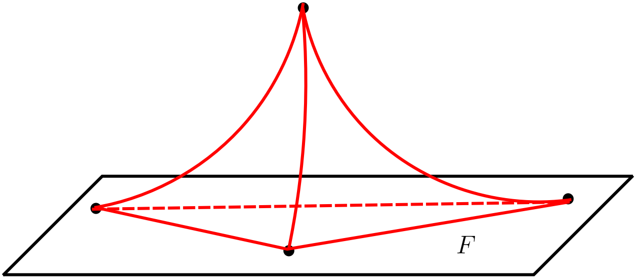
A possibly bad simplex

We find it hard to control the total volume of the following simplex because of the codimension-1 face in F:



A possibly bad simplex

We find it hard to control the total volume of the following simplex because of the codimension-1 face in F:



Quick take-away:

- Support of the volume should be away from N.
- Vertices of the desired simplices should be away from N.

Volume and good simplices

Choice of volume: $\rho(x)d\mathrm{vol}_M$, where $\rho(x)$ is supported in $\{p \in M, d(p,N) \in (\epsilon_5/4, \epsilon_5/2)\}$ for some EXTREEEEEEEEEEEEEEMLY small $\epsilon_5 > 0$.

Volume and good simplices

Choice of volume: $\rho(x)d\mathrm{vol}_M$, where $\rho(x)$ is supported in $\{p \in M, d(p, N) \in (\epsilon_5/4, \epsilon_5/2)\}$ for some EXTREEEEEEEEEEEEEMLY small $\epsilon_5 > 0$.
Good simplices: Let $x_0 \in X$ be a fixed point such that $0 < d(x_0, F) = \epsilon_0 \ll 1$, a simplex is *good* if its vertices are in Γx_0.

Volume and good simplices

Choice of volume: $\rho(x)d\mathrm{vol}_M$, where $\rho(x)$ is supported in $\{p \in M, d(p,N) \in (\epsilon_5/4, \epsilon_5/2)\}$ for some EXTREEEEEEEEEEEEEMLY small $\epsilon_5 > 0$.
Good simplices: Let $x_0 \in X$ be a fixed point such that $0 < d(x_0, F) = \epsilon_0 \ll 1$, a simplex is *good* if its vertices are in Γx_0.

Lemma

Let σ be a good geodesic (or a good simplex obtained by a suitable barycentric construction) of dimension k in X with vertices $p_0, \dots, p_k$. For any $0 < r < R$, we define $A_{r,R}(F) := \{q \in X | d(x,F) \in [r,R]\}$. Then there exists $C = C(r, R, \epsilon_0) > 0$ such that

$$\mathrm{Im}(\sigma) \cap A_{r,R}(F) \subset \bigcup_{j=0}^{k} B_C(\mathrm{Proj}_F(p_j)).$$

Review: barycentric straightening (Lafont-Schmidt, Connell-Wang)

Barycentric construction: Let X, Y be Riemannian manifolds. If there is a smooth function $\Phi : X \times Y \to \mathbb{R}$ such that $\Phi(\cdot; y)$ is strictly convex for any $y \in Y$ which always admits a critical point, then one can construct the corresponding *barycenter map* $\sigma : Y \to X$ by

$$\sigma(y) := \text{the unique critical point of } \Phi(\cdot; y).$$

Barycentric simplices: $Y = \Delta$.

Examples

Let $\sigma : \Delta \to X$ be a singular simplex with vertices $p_0, \ldots, p_k$. If X is simply-connected and nonpositively curved, one can define the straightened simplex $\mathrm{st}(\sigma)$ as a barycenter map using the following Φ.

- $\Phi(q; a_0, \ldots, a_k) = a_0(d(q, p_0))^2 + \ldots a_k(d(q, p_k))^2$. (Theorem A)
- $\Phi(q; a_0, \ldots, a_k) = a_0\mathcal{B}(q, p_0) + \ldots a_k\mathcal{B}(q, p_k)$, where $\mathcal{B}$ is a weighted average of Busemann functions. (Besson-Courtois-Gallot, LS, CW)

Geodesic simplices versus barycentric simplices

Lemma

Let σ be a good geodesic (or a good barycentric simplex) of dimension k in X with vertices $p_0, \ldots, p_k$. For any $0 < r < R$, we define $A_{r,R}(F) := \{q \in X | d(x, F) \in [r, R]\}$. Then there exists $C = C(r, R, \epsilon_0) > 0$ such that

$$\mathrm{Im}(\sigma) \cap A_{r,R}(F) \subset \bigcup_{j=0}^{k} B_C(\mathrm{Proj}_F(p_j)).$$

Assume that σ is top-dimensional. Then,

Good geodesic simplex: $\int_\sigma \chi_{A_{r,R}(F)}(x) d\mathrm{vol}_X(x) \leq ??????$.

Good barycentric simplex: $\int_\sigma \chi_{A_{r,R}(F)}(x) d\mathrm{vol}_X(x) \leq C(r, R, \epsilon_0)$.

Geodesic simplices versus barycentric simplices

Lemma

Let σ be a good geodesic (or a good barycentric simplex) of dimension k in X with vertices $p_0, \ldots, p_k$. For any $0 < r < R$, we define $A_{r,R}(F) := \{q \in X | d(x, F) \in [r, R]\}$. Then there exists $C = C(r, R, \epsilon_0) > 0$ such that

$$\mathrm{Im}(\sigma) \cap A_{r,R}(F) \subset \bigcup_{j=0}^{k} B_C(\mathrm{Proj}_F(p_j)).$$

Assume that σ is top-dimensional. Then,
Good geodesic simplex: $\int_\sigma \chi_{A_{r,R}(F)}(x) d\mathrm{vol}_X(x) \leq$??????.
Good barycentric simplex: $\int_\sigma \chi_{A_{r,R}(F)}(x) d\mathrm{vol}_X(x) \leq C(r, R, \epsilon_0)$.

Barycentric simplices WIN!

Straightened simplices

Special simplices: Any lift of a special simplex σ is a special barycentric simplex. Moreover, at most N elements in ΓF is ϵ_5 close to any lift of it.

Straightened simplices

Special simplices: Any lift of a special simplex σ is a special barycentric simplex. Moreover, at most N elements in ΓF is ϵ_5 close to any lift of it.
Straightened simplices: A weighted sum of special simplices whose total weight is at most C.

Straightened simplices

Special simplices: Any lift of a special simplex σ is a special barycentric simplex. Moreover, at most N elements in ΓF is ϵ_5 close to any lift of it.
Straightened simplices: A weighted sum of special simplices whose total weight is at most C.
Then for any top dimensional simplex σ in M, its straightened simplex $\mathrm{st}(\sigma)$ satisfies

$$\int_{\mathrm{st}(\sigma)} \rho(x) d\mathrm{vol}_M \leq C \cdot NC'(\epsilon_5, \epsilon_0).$$

Straightened simplices

Special simplices: Any lift of a special simplex σ is a special barycentric simplex. Moreover, at most N elements in ΓF is ϵ_5 close to any lift of it.
Straightened simplices: A weighted sum of special simplices whose total weight is at most C.
Then for any top dimensional simplex σ in M, its straightened simplex $\mathrm{st}(\sigma)$ satisfies

$$\int_{\mathrm{st}(\sigma)} \rho(x) d\mathrm{vol}_M \leq C \cdot NC'(\epsilon_5, \epsilon_0).$$

How to construct straightened simplices:

- Start with a good barycentric simplex.
- "Cut" the 1-skeleton of the large good simplex into smaller pieces using ΓF. (Get special simplices.)
- Modify edges using bicombing. (Controlling the total weight.)
- Refill the new 1-skeleton into the straightened simplices.

Table of Contents

Separating simplices by "flats"

Setting: Let σ be a barycentric simplex in X with vertex set $V \subset \Gamma x_0$. In particular, the 1-skeleton of σ is the collection of geodesic segments joining pairs of vertices in V.

Separating simplices by "flats"

Setting: Let σ be a barycentric simplex in X with vertex set $V \subset \Gamma x_0$. In particular, the 1-skeleton of σ is the collection of geodesic segments joining pairs of vertices in V.

Observation 1: If $\widehat{F} \in \Gamma F$ intersects the 1-skeleton of σ, then there exists $\emptyset \neq I \subsetneq V$ such that

$$d([x,y],\widehat{F}) = 0, \ \forall x \in I, y \in V \setminus I.$$

Separating simplices by "flats"

Setting: Let σ be a barycentric simplex in X with vertex set $V \subset \Gamma x_0$. In particular, the 1-skeleton of σ is the collection of geodesic segments joining pairs of vertices in V.

Observation 1: If $\widehat{F} \in \Gamma F$ intersects the 1-skeleton of σ, then there exists $\emptyset \neq I \subsetneq V$ such that

$$d([x, y], \widehat{F}) = 0, \ \forall x \in I, y \in V \setminus I.$$

Observation 2: The above choice of the unordered pair $\{I, V \setminus I\}$ is unique. In other words, for any distinct vertices $x, y \in V$

$$d([x, y], \widehat{F}) = 0 \text{ iff } |\{x, y\} \cap V| = 1.$$

Separating simplices by "flats"

Setting: Let σ be a barycentric simplex in X with vertex set $V \subset \Gamma x_0$. In particular, the 1-skeleton of σ is the collection of geodesic segments joining pairs of vertices in V.

Observation 1: If $\widehat{F} \in \Gamma F$ intersects the 1-skeleton of σ, then there exists $\emptyset \neq I \subsetneq V$ such that

$$d([x,y],\widehat{F}) = 0, \ \forall x \in I, y \in V \setminus I.$$

Observation 2: The above choice of the unordered pair $\{I, V \setminus I\}$ is unique. In other words, for any distinct vertices $x, y \in V$

$$d([x,y],\widehat{F}) = 0 \text{ iff } |\{x,y\} \cap V| = 1.$$

Observation 3: Let $W \subset V$. The W-face of σ is the face of σ whose vertex set is exactly W.

Separating simplices by "flats"

Setting: Let σ be a barycentric simplex in X with vertex set $V \subset \Gamma x_0$. In particular, the 1-skeleton of σ is the collection of geodesic segments joining pairs of vertices in V.

Observation 1: If $\widehat{F} \in \Gamma F$ intersects the 1-skeleton of σ, then there exists $\emptyset \neq I \subsetneq V$ such that

$$d([x,y], \widehat{F}) = 0, \ \forall x \in I, y \in V \setminus I.$$

Observation 2: The above choice of the unordered pair $\{I, V \setminus I\}$ is unique. In other words, for any distinct vertices $x, y \in V$

$$d([x,y], \widehat{F}) = 0 \text{ iff } |\{x,y\} \cap V| = 1.$$

Observation 3: Let $W \subset V$. The W-face of σ is the face of σ whose vertex set is exactly W.If in addition that $\widehat{F}$ intersects the 1-skeleton of σ, then for any distinct vertices $x, y \in W$

$$d([x,y], \widehat{F}) = 0 \text{ iff } |\{x,y\} \cap W| = 1.$$

Actual separation

Actural separation: We say that $\widehat{F} \in \Gamma F$ *(actually) separates* σ if $\widehat{F}$ intersects the 1-skeleton of σ.

Actual separation

Actural separation: We say that $\widehat{F} \in \Gamma F$ *(actually) separates* σ if $\widehat{F}$ intersects the 1-skeleton of σ.

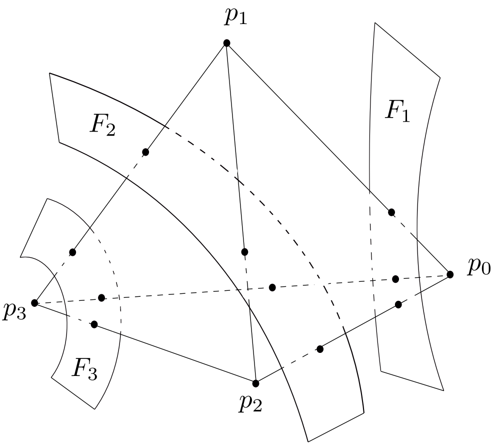
Separation type: Let $\emptyset \neq I \subset V$. We say that the unordered pair $\{I, V \setminus I\}$ is an (actual) separation type for $\widehat{F}$ w.r.t σ if

$$d([x, y], \widehat{F}) = 0, \ \forall x \in I, y \in V \setminus I.$$

Properties:

- (**Ob. 2**) Actual separation types are unique.
- (**Ob. 3**) Let $\mathrm{Res}_W^V(\{I, V \setminus I\}) := \{I \cap W, W \setminus I\}$. If $\{I, V \setminus I\}$ is an actual separation type for $\widehat{F}$ w.r.t σ, then $\mathrm{Res}_W^V(\{I, V \setminus I\})$ is also an actual separation type for $\widehat{F}$ w.r.t the W-face of σ.
- (**Ob. 4**) Let $F_1, F_2 \in \Gamma F$ be distinct elements with separation types $\{I_1, V \setminus I_1\}$, $\{I_2, V \setminus I_2\}$ respectively. Then WLOG, we can assume that $I_1 \cap I_2 = \emptyset$.

Observation 4: Let $F_1, F_2 \in \Gamma F$ be distinct elements with separation types $\{I_1, V \setminus I_1\}$, $\{I_2, V \setminus I_2\}$ respectively. Then WLOG, we can assume that $I_1 \cap I_2 = \emptyset$.



Problem with actual separation

A Review on our constructions in the straightening arguments:

- **Volume**: $\rho(x) d\mathrm{vol}_M$, where $\rho(x)$ is supported in $\{p \in M, d(p, N) \in (\epsilon_5/4, \epsilon_5/2)\}$ for some EXTREEEEEEEEEEEEEMLY small $\epsilon_5 > 0$.
- **Special simplices**: Any lift of a special simplex σ is a special barycentric simplex. Moreover, at most N elements in ΓF is ϵ_5 close to any lift of it.
- **Straightened simplices**: A weighted sum of special simplices whose total weight is at most C.

A Review on our constructions in the straightening arguments:

- **Volume**: $\rho(x)d\mathrm{vol}_M$, where $\rho(x)$ is supported in $\{p \in M, d(p, N) \in (\epsilon_5/4, \epsilon_5/2)\}$ for some EXTREEEEEEEEEEEEEMLY small $\epsilon_5 > 0$.
- **Special simplices**: Any lift of a special simplex σ is a special barycentric simplex. Moreover, at most N elements in ΓF is ϵ_5 close to any lift of it.
- **Straightened simplices**: A weighted sum of special simplices whose total weight is at most C.

Problem: If a "flat" does not intersects a simplex but is close to the simplex, it contributes to volume. We should consider these "flats" as well!

Almost separation

Definition (ϵ-almost separation)

We say that $\widehat{F} \in \Gamma F$ *ϵ-almost separates* σ if there exists $\emptyset \neq I \subset V$ such that

$$d([x,y],\widehat{F}) \leq \epsilon, \ \forall x \in I, y \in V \setminus I.$$

$\{I, V \setminus I\}$ is called an *ϵ-almost separation type* of $\widehat{F}$ w.r.t σ.
The collection of all separation types of $\widehat{F}$ with respect to σ is denoted as $\mathrm{Sep}_V(F)$.

Almost separation

Definition (ϵ-almost separation)

We say that $\widehat{F} \in \Gamma F$ *ϵ-almost separates* σ if there exists $\emptyset \neq I \subset V$ such that

$$d([x,y], \widehat{F}) \leq \epsilon, \ \forall x \in I, y \in V \setminus I.$$

$\{I, V \setminus I\}$ is called an *ϵ-almost separation type* of $\widehat{F}$ w.r.t σ.
The collection of all separation types of $\widehat{F}$ with respect to σ is denoted as $\mathrm{Sep}_V(F)$.

Remark: When $\epsilon = 0$, this is the same as actual separation.

Properties of actual separation

Properties of actual separation:

- (**Ob. 2**) Actual separation types are unique.
- (**Ob. 3**) Let $\mathrm{Res}^V_W(\{I, V \setminus I\}) := \{I \cap W, W \setminus I\}$. If $\{I, V \setminus I\}$ is an actual separation type for $\widehat{F}$ w.r.t σ, then $\mathrm{Res}^V_W(\{I, V \setminus I\})$ is also an actual separation type for $\widehat{F}$ w.r.t the W-face of σ.
- (**Ob. 4**) Let $F_1, F_2 \in \Gamma F$ be distinct elements with separation types $\{I_1, V \setminus I_1\}$, $\{I_2, V \setminus I_2\}$ respectively. Then WLOG, we can assume that $I_1 \cap I_2 = \emptyset$.

Properties of almost separation:

- (**Ob. 2**) Almost separation types are NOT unique.
- (**Ob. 3**) Let $\mathrm{Res}^V_W(\{I, V \setminus I\}) := \{I \cap W, W \setminus I\}$. If $\{I, V \setminus I\}$ is an ϵ-almost separation type for $\widehat{F}$ w.r.t σ, then $\mathrm{Res}^V_W(\{I, V \setminus I\})$ is also an ϵ-almost separation type for $\widehat{F}$ w.r.t the W-face of σ.
- (**Ob. 4**) Assume that $\epsilon < \epsilon_0 \ll 1$. Let $F_1, F_2 \in \Gamma F$ be distinct elements with ϵ-almost separation types $\{I_1, V \setminus I_1\}$, $\{I_2, V \setminus I_2\}$ respectively. Then WLOG, we can assume that $I_1 \cap I_2 = \emptyset$.

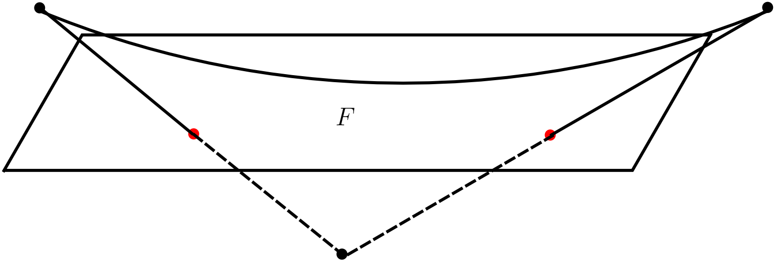
Properties of almost separation

Properties of almost separation:

- (**Ob. 2**) Almost separation types are NOT unique.
- (**Ob. 3**) Let $\mathrm{Res}^V_W(\{I, V \setminus I\}) := \{I \cap W, W \setminus I\}$. If $\{I, V \setminus I\}$ is an ϵ-almost separation type for $\widehat{F}$ w.r.t σ, then $\mathrm{Res}^V_W(\{I, V \setminus I\})$ is also an ϵ-almost separation type for $\widehat{F}$ w.r.t the W-face of σ.
- (**Ob. 4**) Assume that $\epsilon < \epsilon_0 \ll 1$. Let $F_1, F_2 \in \Gamma F$ be distinct elements with ϵ-almost separation types $\{I_1, V \setminus I_1\}$, $\{I_2, V \setminus I_2\}$ respectively. Then WLOG, we can assume that $I_1 \cap I_2 = \emptyset$.
- For any $\epsilon > 0$, there exists some constant $c(\epsilon) \in (0, \epsilon)$ such that if $\widehat{F}$ does not ϵ-almost separate σ, then $d(\sigma, \widehat{F}) \geq c(\epsilon)$. **Therefore, the volume of σ is only related to "flats" which almost separates it!**

Possible bizarre phenomena

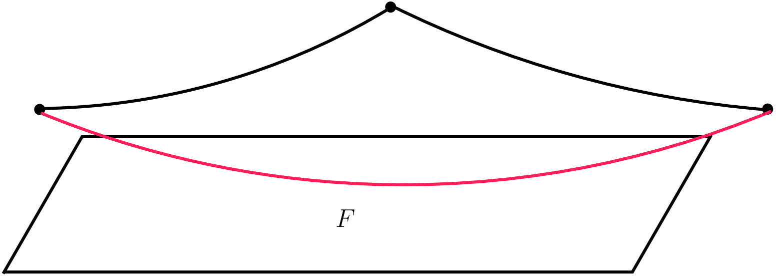
Multiple almost separation types:



All three edges are ϵ-close to F. **This makes our notations and definitions more complicated. However, this does not cause any essential trouble for us.**

Possible bizarre phenomena

"Flats" ϵ-almost intersecting an edge but not ϵ-almost intersecting the simplex.



Only one edge is ϵ-close to F. **This and its variants are the major threat when we prove Theorem A!**

Possible bizarre phenomena

Edges getting ϵ-almost separated, but not revealed by separation types: Let x, y be distinct vertices in V. The following is possible:

- $\widehat{F}$ ϵ-almost separate $[x, y]$ and σ.
- For any ϵ-almost separation type $\{I, V \setminus I\}$, either $x, y \in I$ or $x, y \in V \setminus I$.

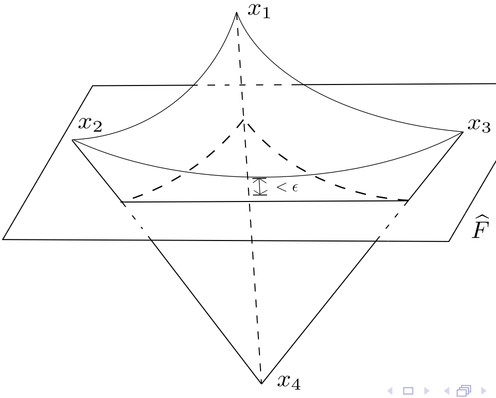


Table of Contents

Mineyev's bicombing construction revisited

Key steps:

- **Step 1**: Cut long geodesics.
- **Step 2**: Construct $f(x, y)$ inductively. (This step is the key to control $|\cdot|_{l^1}$ of the boundary of a 2-simplex.)
- **Step 3**: Use $f(x, y)$ or $\overline{f}(x, y)$ to construct $\beta'(x, y)$.
- **Step 4**: Anti-symmetrize.

Mineyev's bicombing construction revisited

Key steps:

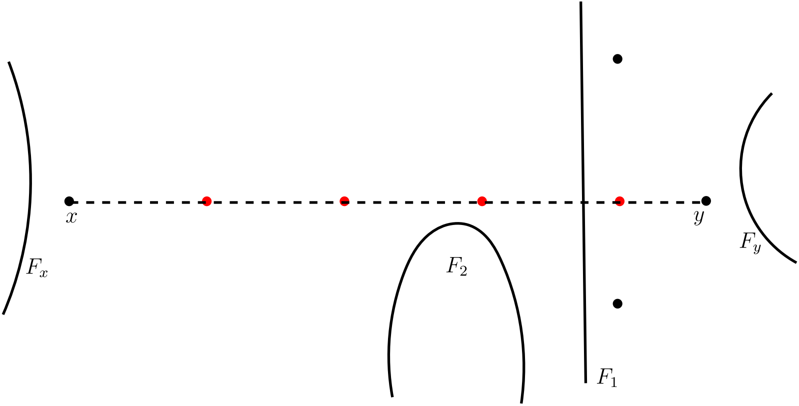
- **Step 1**: Cut long geodesics.
- **Step 2**: Construct $f(x, y)$ inductively. (This step is the key to control $|\cdot|_{l^1}$ of the boundary of a 2-simplex.)
- **Step 3**: Use $f(x, y)$ or $\overline{f}(x, y)$ to construct $\beta'(x, y)$.
- **Step 4**: Anti-symmetrize.

Main challenges:

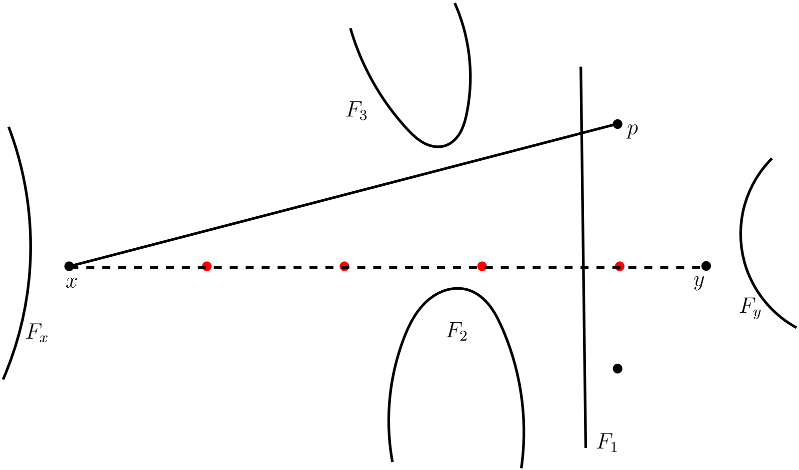
- **Step 1**: How to cut?
- **Step 2**: How to construct $f(x, y)$ and ensure enough cancellation?
- **Step 3** and **Step 4** are not problematic for us.

Cutting long edges

Natural cutting: Cut a long geodesic using "flats" that are ϵ-almost separating the edge.



Problem with ϵ-almost separation



Problem with ϵ-almost separation

Problem: In the picture, the cutting along $[x, p]$ may be unrelated to the cutting along $[x, y]$.

Problem with ϵ-almost separation

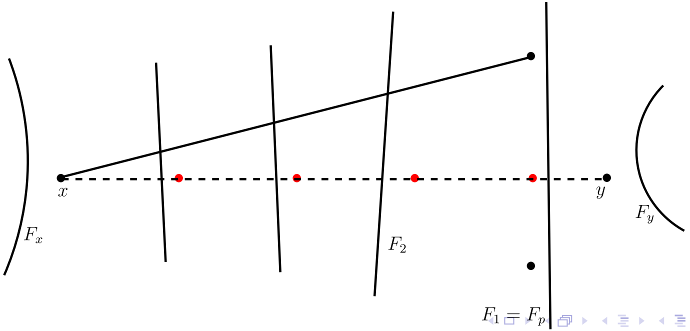
Problem: In the picture, the cutting along $[x, p]$ may be unrelated to the cutting along $[x, y]$.
Good news: Vertices of geodesic segments that we care about are all close to a unique "flat".

Problem with ϵ-almost separation

Problem: In the picture, the cutting along $[x, p]$ may be unrelated to the cutting along $[x, y]$.

Good news: Vertices of geodesic segments that we care about are all close to a unique "flat".

Resolving the issue: We cut the geodesic $[x, y]$ only by "flats" that are between F_x and F_y. This notion of "in-between" must satisfy some "linear ordering". (To be explained in the next slide.)



In-between for "flats"

"In-between": For any $\widehat{F}, F_1, F_2 \in \Gamma F$, $\widehat{F}$ is between F_1 and F_2 if either of the following holds:

- $\widehat{F} = F_1$ or F_2.
- F_1 and F_2 are on two different sides of $\widehat{F}$.

In-between for "flats"

"In-between": For any $\widehat{F}, F_1, F_2 \in \Gamma F$, $\widehat{F}$ is between F_1 and F_2 if either of the following holds:

- $\widehat{F} = F_1$ or F_2.
- F_1 and F_2 are on two different sides of $\widehat{F}$.

Properties of "in-between" for real numbers: let a, b be real numbers.

- If c is between a and b, then any real number d which is between a and c is also between a and b
- If c is between a and b, then c is the real number which is simultaneously between a and c and between c and b.

Properties of "in-between": (Linear ordering properties)

- If F_3 is between F_1 and F_2, then any element $\widehat{F} \in \Gamma F$ which is between F_1 and F_3 is also between F_1 and F_2
- If F_3 is between F_1 and F_2, then F_3 is the unique element in ΓF which is simultaneously between F_1 and F_3 and between F_3 and F_2.

Problems with ϵ-almost in-between

Attempt 1: For any $\widehat{F}, F_1, F_2 \in \Gamma F$, $\widehat{F}$ is between F_1 and F_2 if either of the following holds:

- $\widehat{F} = F_1$ or F_2.
- There exists some p_i near F_i such that $\widehat{F}$ ϵ-almost separates $[p_1, p_2]$.

Problems with ϵ-almost in-between

Attempt 1: For any $\widehat{F}, F_1, F_2 \in \Gamma F$, $\widehat{F}$ is between F_1 and F_2 if either of the following holds:

- $\widehat{F} = F_1$ or F_2.
- There exists some p_i near F_i such that $\widehat{F}$ ϵ-almost separates $[p_1, p_2]$.

Attempt 2: For any $\widehat{F}, F_1, F_2 \in \Gamma F$, $\widehat{F}$ is between F_1 and F_2 if either of the following holds:

- $\widehat{F} = F_1$ or F_2.
- For any p_i near F_i, $i = 1, 2$, the element $\widehat{F}$ ϵ-almost separates $[p_1, p_2]$.

Problems with ϵ-almost in-between

Attempt 1: For any $\widehat{F}, F_1, F_2 \in \Gamma F$, $\widehat{F}$ is between F_1 and F_2 if either of the following holds:

- $\widehat{F} = F_1$ or F_2.
- There exists some p_i near F_i such that $\widehat{F}$ ϵ-almost separates $[p_1, p_2]$.

Attempt 2: For any $\widehat{F}, F_1, F_2 \in \Gamma F$, $\widehat{F}$ is between F_1 and F_2 if either of the following holds:

- $\widehat{F} = F_1$ or F_2.
- For any p_i near F_i, $i = 1, 2$, the element $\widehat{F}$ ϵ-almost separates $[p_1, p_2]$.

Problem: It is hard to tell whether linear ordering properties hold for any of the above definition of "ϵ-almost in-between".

Improved notion of “almost in between”: $\Omega(F_1, F_2)$

Recall that for any $x \in \Gamma x_0$, $d(x, F_x) \leq \epsilon_0 \ll 1$. Choose $0 < \epsilon_2 \ll \epsilon_1 \ll \epsilon_0$.

Improved notion of "almost in between": $\Omega(F_1, F_2)$

Recall that for any $x \in \Gamma x_0$, $d(x, F_x) \leq \epsilon_0 \ll 1$. Choose $0 < \epsilon_2 \ll \epsilon_1 \ll \epsilon_0$.

Definition of $\Omega(\cdot, \cdot)$

For any $F_1, F_2 \in \Gamma F$, we define $\Omega_0(F_1, F_2) \subset \Gamma F$ such that

$$\Omega_0(F_1, F_2) = \left\{ \widehat{F} \in \Gamma F \;\middle|\; \begin{array}{l} \exists p_j \in X \text{ s.t. } d(p_j, F_j) \leq \epsilon_0, j = 1, 2, \\ \text{and } d([p_1, p_2], \widehat{F}) < \epsilon_2. \end{array} \right\}.$$

Improved notion of "almost in between": $\Omega(F_1, F_2)$

Recall that for any $x \in \Gamma x_0$, $d(x, F_x) \leq \epsilon_0 \ll 1$. Choose $0 < \epsilon_2 \ll \epsilon_1 \ll \epsilon_0$.

Definition of $\Omega(\cdot, \cdot)$

For any $F_1, F_2 \in \Gamma F$, we define $\Omega_0(F_1, F_2) \subset \Gamma F$ such that

$$\Omega_0(F_1, F_2) = \left\{ \widehat{F} \in \Gamma F \;\middle|\; \begin{array}{l} \exists p_j \in X \text{ s.t. } d(p_j, F_j) \leq \epsilon_0, j = 1, 2, \\ \text{and } d([p_1, p_2], \widehat{F}) < \epsilon_2. \end{array} \right\}.$$

Inductively we define

$$\Omega_k(F_1, F_2) := \bigcup_{F', F'' \in \Omega_{k-1}(F_1, F_2)} \Omega_0(F', F'').$$

Improved notion of "almost in between": $\Omega(F_1, F_2)$

Recall that for any $x \in \Gamma x_0$, $d(x, F_x) \leq \epsilon_0 \ll 1$. Choose $0 < \epsilon_2 \ll \epsilon_1 \ll \epsilon_0$.

Definition of $\Omega(\cdot, \cdot)$

For any $F_1, F_2 \in \Gamma F$, we define $\Omega_0(F_1, F_2) \subset \Gamma F$ such that

$$\Omega_0(F_1, F_2) = \left\{ \widehat{F} \in \Gamma F \;\middle|\; \begin{array}{l} \exists p_j \in X \text{ s.t. } d(p_j, F_j) \leq \epsilon_0, j = 1, 2, \\ \text{and } d([p_1, p_2], \widehat{F}) < \epsilon_2. \end{array} \right\}.$$

Inductively we define

$$\Omega_k(F_1, F_2) := \bigcup_{F', F'' \in \Omega_{k-1}(F_1, F_2)} \Omega_0(F', F'').$$

Finally, we define

$$\Omega(F_1, F_2) := \bigcup_{j=0}^{\infty} \Omega_k(F_1, F_2).$$

Properties of $\Omega(\cdot,\cdot)$: (Think about $\Omega(F_1, F_2)$ as those "flats" which are almost between F_1 and F_2.)

Properties of $\Omega(\cdot,\cdot)$

Properties of $\Omega(\cdot,\cdot)$: (Think about $\Omega(F_1, F_2)$ as those "flats" which are almost between F_1 and F_2.)

- For any $\widehat{F} \in \Omega(F_1, F_2) \setminus \{F_1, F_2\}$ and any x_1, x_2 such that $d(x_i, F_i) < \epsilon_0$, $\widehat{F}$ ϵ_1-almost separates $[x_1, x_2]$.

Properties of $\Omega(\cdot, \cdot)$

Properties of $\Omega(\cdot, \cdot)$: (Think about $\Omega(F_1, F_2)$ as those "flats" which are almost between F_1 and F_2.)

- For any $\widehat{F} \in \Omega(F_1, F_2) \setminus \{F_1, F_2\}$ and any x_1, x_2 such that $d(x_i, F_i) < \epsilon_0$, $\widehat{F}$ ϵ_1-almost separates $[x_1, x_2]$.
- For any $F_3 \in \Omega(F_1, F_2)$, $\Omega(F_1, F_3) \subset \Omega(F_1, F_2)$ and $\Omega(F_3, F_2) \subset \Omega(F_1, F_2)$.
- For any $F_3 \in \Omega(F_1, F_2)$, we have $\Omega(F_1, F_3) \cap \Omega(F_3, F_2) = \{F_3\}$.

Properties of $\Omega(\cdot,\cdot)$

Properties of $\Omega(\cdot,\cdot)$: (Think about $\Omega(F_1,F_2)$ as those "flats" which are almost between F_1 and F_2.)

- For any $\widehat{F} \in \Omega(F_1,F_2) \setminus \{F_1,F_2\}$ and any x_1,x_2 such that $d(x_i,F_i) < \epsilon_0$, $\widehat{F}$ ϵ_1-almost separates $[x_1,x_2]$.
- For any $F_3 \in \Omega(F_1,F_2)$, $\Omega(F_1,F_3) \subset \Omega(F_1,F_2)$ and $\Omega(F_3,F_2) \subset \Omega(F_1,F_2)$.
- For any $F_3 \in \Omega(F_1,F_2)$, we have $\Omega(F_1,F_3) \cap \Omega(F_3,F_2) = \{F_3\}$.

Properties of "in-between": (Linear ordering properties)

- If F_3 is between F_1 and F_2, then any element $\widehat{F} \in \Gamma F$ which is between F_1 and F_3 is also between F_1 and F_2
- If F_3 is between F_1 and F_2, then F_3 is the unique element in ΓF which is simultaneously between F_1 and F_3 and between F_3 and F_2.

Main problem with $\Omega(\cdot, \cdot)$

Key steps:

- **Step 1**: Cut long geodesics.
- **Step 2**: Construct $f(x, y)$ inductively. (This step is the key to control $|\cdot|_{l^1}$ of the boundary of a 2-simplex.)
- **Step 3**: Use $f(x, y)$ or $\overline{f}(x, y)$ to construct $\beta'(x, y)$.
- **Step 4**: Anti-symmetrize.

Main problem with $\Omega(\cdot,\cdot)$

Key steps:

- **Step 1**: Cut long geodesics.
- **Step 2**: Construct $f(x,y)$ inductively. (This step is the key to control $|\cdot|_{l^1}$ of the boundary of a 2-simplex.)
- **Step 3**: Use $f(x,y)$ or $\overline{f}(x,y)$ to construct $\beta'(x,y)$.
- **Step 4**: Anti-symmetrize.

Main challenges:

- **Step 1**: How to cut? (Resolved by $\Omega(\cdot,\cdot)$)
- **Step 2**: How to construct $f(x,y)$ and ensure enough cancellation?
- **Step 3** and **Step 4** are not problematic for us.

Main problem with $\Omega(\cdot,\cdot)$

Key steps:

- **Step 1**: Cut long geodesics.
- **Step 2**: Construct $f(x,y)$ inductively. (This step is the key to control $|\cdot|_{l^1}$ of the boundary of a 2-simplex.)
- **Step 3**: Use $f(x,y)$ or $\overline{f}(x,y)$ to construct $\beta'(x,y)$.
- **Step 4**: Anti-symmetrize.

Main challenges:

- **Step 1**: How to cut? (Resolved by $\Omega(\cdot,\cdot)$)
- **Step 2**: How to construct $f(x,y)$ and ensure enough cancellation?
- **Step 3** and **Step 4** are not problematic for us.

Problem with Ω:
One can perform **Step 2** as well with $\Omega(\cdot,\cdot)$, but it is unclear cancellations are enough.

Improving notion of "almost in between": $\Theta(F_1, F_2)$

Definition of $\Theta(\cdot,\cdot)$

Let $0 < \epsilon_4 \ll \epsilon_3 \ll \epsilon_2$. For any $F_1, F_2 \in \Gamma F$, we define

$$\Theta_0(F_1, F_2) = \{F_1, F_2\} \cup \left\{ \widehat{F} \in \Gamma F \;\middle|\; \begin{array}{l} \exists p_j' \in X \text{ and } F_j' \in \Omega(F_1, F_2) \\ \text{s.t. } \widehat{F} \neq F_j', d(p_j', F_j') \leq \epsilon_0, j = 1, 2, \\ \text{ and } d([p_1', p_2'], \widehat{F}) < \epsilon_4 \end{array} \right\}.$$

Similar to $\Omega(\cdot,\cdot)$, we define

$$\Theta_k(F_1, F_2) := \bigcup_{F', F'' \in \Theta_{k-1}(F_1, F_2)} \Theta_0(F', F'').$$

Finally, we define

$$\Theta(F_1, F_2) := \bigcup_{j=0}^{\infty} \Theta_k(F_1, F_2).$$

Properties of $\Theta(\cdot, \cdot)$

Properties of $\Theta(\cdot, \cdot)$ (similar to those of $\Omega(\cdot, \cdot)$): (Think about $\Theta(F_1, F_2)$ as those "flats" which are almost between F_1 and F_2.)

Properties of $\Theta(\cdot, \cdot)$

Properties of $\Theta(\cdot, \cdot)$ (similar to those of $\Omega(\cdot, \cdot)$): (Think about $\Theta(F_1, F_2)$ as those "flats" which are almost between F_1 and F_2.)

- For any $\widehat{F} \in \Theta(F_1, F_2) \setminus \{F_1, F_2\}$ and any x_1, x_2 such that $d(x_i, F_i) < \epsilon_0$, $\widehat{F}$ ϵ_3-almost separates $[x_1, x_2]$.

Properties of $\Theta(\cdot,\cdot)$ (similar to those of $\Omega(\cdot,\cdot)$): (Think about $\Theta(F_1, F_2)$ as those "flats" which are almost between F_1 and F_2.)

- For any $\widehat{F} \in \Theta(F_1, F_2) \setminus \{F_1, F_2\}$ and any x_1, x_2 such that $d(x_i, F_i) < \epsilon_0$, $\widehat{F}$ ϵ_3-almost separates $[x_1, x_2]$.
- For any $F_3 \in \Theta(F_1, F_2)$, $\Theta(F_1, F_3) \subset \Theta(F_1, F_2)$ and $\Theta(F_3, F_2) \subset \Theta(F_1, F_2)$.
- For any $F_3 \in \Theta(F_1, F_2)$, we have $\Theta(F_1, F_3) \cap \Theta(F_3, F_2) = \{F_3\}$.

Properties of $\Theta(\cdot,\cdot)$

Properties of $\Theta(\cdot,\cdot)$ (similar to those of $\Omega(\cdot,\cdot)$): (Think about $\Theta(F_1, F_2)$ as those "flats" which are almost between F_1 and F_2.)

- For any $\widehat{F} \in \Theta(F_1, F_2) \setminus \{F_1, F_2\}$ and any x_1, x_2 such that $d(x_i, F_i) < \epsilon_0$, $\widehat{F}$ ϵ_3-almost separates $[x_1, x_2]$.
- For any $F_3 \in \Theta(F_1, F_2)$, $\Theta(F_1, F_3) \subset \Theta(F_1, F_2)$ and $\Theta(F_3, F_2) \subset \Theta(F_1, F_2)$.
- For any $F_3 \in \Theta(F_1, F_2)$, we have $\Theta(F_1, F_3) \cap \Theta(F_3, F_2) = \{F_3\}$.

Additional property of $\Theta(\cdot,\cdot)$: Let
$\mathcal{F}(x,y,z) := \Theta(F_x, F_y) \cup \Theta(F_y, F_z) \cup \Theta(F_z, F_x)$,
$\mathcal{F}_x(y,z) := \Theta(F_x, F_y) \cap \Theta(F_y, F_z)$, $\mathcal{F}_y(z,x) := \Theta(F_y, F_z) \cap \Theta(F_y, F_x)$,
$\mathcal{F}_z(x,y) := \Theta(F_z, F_x) \cap \Theta(F_z, F_y)$ and

$$\mathcal{A}(x,y,z) := \mathcal{F}_x(y,z) \cup \mathcal{F}_y(z,x) \cup \mathcal{F}_z(x,y).$$

Properties of $\Theta(\cdot,\cdot)$ (similar to those of $\Omega(\cdot,\cdot)$): (Think about $\Theta(F_1, F_2)$ as those "flats" which are almost between F_1 and F_2.)

- For any $\widehat{F} \in \Theta(F_1, F_2) \setminus \{F_1, F_2\}$ and any x_1, x_2 such that $d(x_i, F_i) < \epsilon_0$, $\widehat{F}$ ϵ_3-almost separates $[x_1, x_2]$.
- For any $F_3 \in \Theta(F_1, F_2)$, $\Theta(F_1, F_3) \subset \Theta(F_1, F_2)$ and $\Theta(F_3, F_2) \subset \Theta(F_1, F_2)$.
- For any $F_3 \in \Theta(F_1, F_2)$, we have $\Theta(F_1, F_3) \cap \Theta(F_3, F_2) = \{F_3\}$.

Additional property of $\Theta(\cdot,\cdot)$: Let
$\mathcal{F}(x,y,z) := \Theta(F_x, F_y) \cup \Theta(F_y, F_z) \cup \Theta(F_z, F_x)$,
$\mathcal{F}_x(y,z) := \Theta(F_x, F_y) \cap \Theta(F_y, F_z)$, $\mathcal{F}_y(z,x) := \Theta(F_y, F_z) \cap \Theta(F_y, F_x)$,
$\mathcal{F}_z(x,y) := \Theta(F_z, F_x) \cap \Theta(F_z, F_y)$ and

$$\mathcal{A}(x,y,z) := \mathcal{F}_x(y,z) \cup \mathcal{F}_y(z,x) \cup \mathcal{F}_z(x,y).$$

Then

$$|\mathcal{F}(x,y,z) \setminus \mathcal{A}(x,y,z)| \le 3.$$

Interpreting the additional property

- If we replace Ω with Θ, the bicombing construction works completely with the desired norm control.

Interpreting the additional property

- If we replace Ω with Θ, the bicombing construction works completely with the desired norm control.
- Consider the following graph: vertices are ΓF; two distinct vertices F_1 and F_2 are joined by an edge if and only if $\Theta(F_1, F_2) = \{F_1, F_2\}$.

Interpreting the additional property

- If we replace Ω with Θ, the bicombing construction works completely with the desired norm control.
- Consider the following graph: vertices are ΓF; two distinct vertices F_1 and F_2 are joined by an edge if and only if $\Theta(F_1, F_2) = \{F_1, F_2\}$. Then the graph is Gromov hyperbolic.

Interpreting the additional property

- If we replace Ω with Θ, the bicombing construction works completely with the desired norm control.
- Consider the following graph: vertices are ΓF; two distinct vertices F_1 and F_2 are joined by an edge if and only if $\Theta(F_1, F_2) = \{F_1, F_2\}$. Then the graph is Gromov hyperbolic.

We can also define the notion of Θ-separation as a notion of almost separation:

Definition (ϵ-almost separation)

We say that $\widehat{F} \in \Gamma F$ *ϵ-almost separates σ* if there exists $\emptyset \neq I \subset V$ such that

$$\widehat{F} \in \Theta(F_x, F_y), \ \forall x \in I, y \in V \setminus I.$$

$\{I, V \setminus I\}$ is called a *Θ-separation type* of $\widehat{F}$ w.r.t σ.
The collection of all separation types of $\widehat{F}$ with respect to σ is denoted as $\mathrm{Sep}_V(F)$.

Interpreting the additional property

- If we replace Ω with Θ, the bicombing construction works completely with the desired norm control.
- Consider the following graph: vertices are ΓF; two distinct vertices F_1 and F_2 are joined by an edge if and only if $\Theta(F_1, F_2) = \{F_1, F_2\}$. Then the graph is Gromov hyperbolic.

We can also define the notion of Θ-separation as a notion of almost separation:

Definition (ϵ-almost separation)

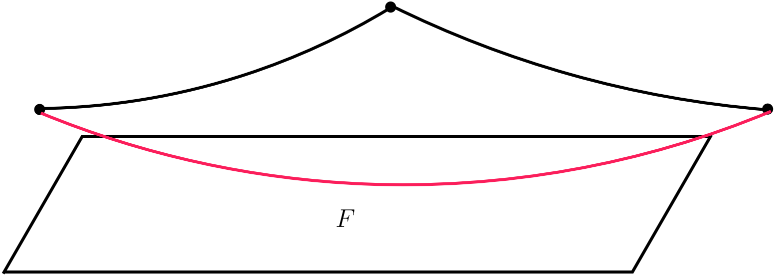
We say that $\widehat{F} \in \Gamma F$ *ϵ-almost separates σ* if there exists $\emptyset \neq I \subset V$ such that

$$\widehat{F} \in \Theta(F_x, F_y), \ \forall x \in I, y \in V \setminus I.$$

$\{I, V \setminus I\}$ is called a *Θ-separation type* of $\widehat{F}$ w.r.t σ.
The collection of all separation types of $\widehat{F}$ with respect to σ is denoted as $\mathrm{Sep}_V(F)$.

Bad things may happen, but......

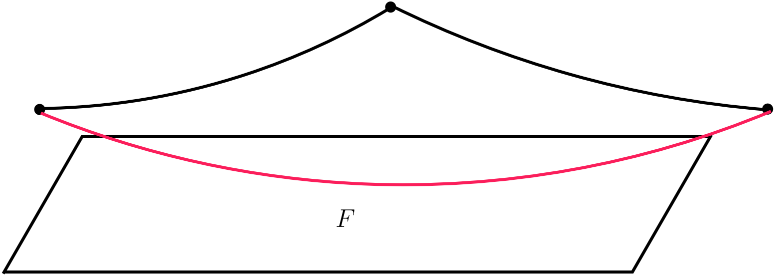
"Flats" Θ-almost intersecting an edge but not Θ-almost intersecting the simplex.



Only one edge is ϵ-close to F. **This and its variants are the major threat when we prove Theorem A!**

Bad things may happen, but......

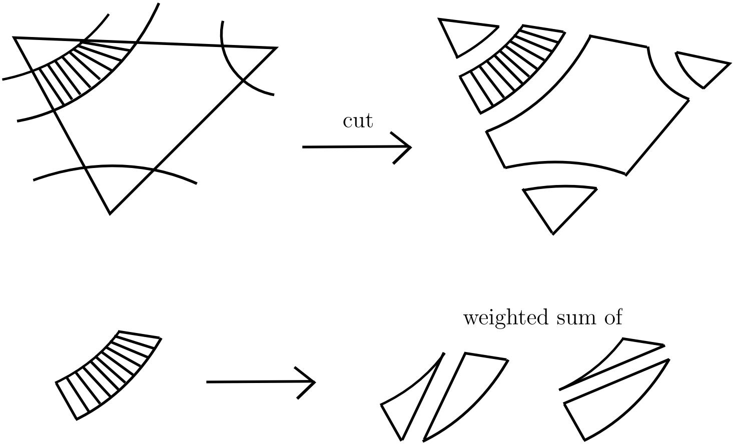
“Flats” Θ-almost intersecting an edge but not Θ-almost intersecting the simplex.



Only one edge is ϵ-close to F. **This and its variants are the major threat when we prove Theorem A!**
The additional property implies that bad Θ-separation only happens finitely many times!

Table of Contents

Cut and triangulate



Story of (actual) separation: graph of "flats"

Let $\mathcal{F}(V)$ be all "flats" in ΓF which actually separates the simplex with vertices V. We also include F_x for any $x \in V$ in $\mathcal{F}(V)$. Assume that

- For any $x \in V$, F_x does not separates the simplex.
- $\{F_x\}_{x \in V}$ are pairwise distinct.

Graph of actual separation: $\mathbf{G}_V$ is a graph with vertices $\mathcal{F}(V)$. Its edges are defined as follows: two distinct F_1 and F_2 are joined by an edge if there are no other "flats" in between them.

Properties of $\mathbf{G}_V$:

- There is a one-to-one correspondence between polygonal chambers of the simplex after actual separation, and maximal complete subgraphs (MCS) of $\mathbf{G}_V$.
- Every vertex is shared by at most 2 MCS.
- Every edge is contained in a unique MCS.
- For any $W \subset V$, W-face of the simplex, $\mathbf{G}_W$ is a subgraph of $\mathbf{G}_V$.

Upgrading to the story of Θ-separation

Troubles:

- Multiple separation types for the same "flat".
- Potential bad phenomena. ("flats" Θ-separating an edge but not the simplex.)

Upgrading to the story of Θ-separation

Troubles:

- Multiple separation types for the same “flat”.
- Potential bad phenomena. (“flats” Θ-separating an edge but not the simplex.)

How to resolve:

- When constructing vertices of $\mathbf{G}_V$, use good “flats” only. Also mark these flats with good separation types. (Same “flat” with different separation types are viewed as different vertices).

Upgrading to the story of Θ-separation

Troubles:

- Multiple separation types for the same "flat".
- Potential bad phenomena. ("flats" Θ-separating an edge but not the simplex.)

How to resolve:

- When constructing vertices of $\mathbf{G}_V$, use good "flats" only. Also mark these flats with good separation types. (Same "flat" with different separation types are viewed as different vertices).
- Use Θ to define edges.

Upgrading to the story of Θ-separation

Troubles:

- Multiple separation types for the same "flat".
- Potential bad phenomena. ("flats" Θ-separating an edge but not the simplex.)

How to resolve:

- When constructing vertices of $\mathbf{G}_V$, use good "flats" only. Also mark these flats with good separation types. (Same "flat" with different separation types are viewed as different vertices).
- Use Θ to define edges.
- For any $W \subset V$, $\mathbf{G}_W$ may contain more information than a corresponding subgraph of $\mathbf{G}_V$. ($\mathbf{G}_W$ is an "enrichment" of a subgraph of $\mathbf{G}_V$.) Tedious technical discussions are needed.

Upgrading to the story of Θ-separation

Troubles:

- Multiple separation types for the same "flat".
- Potential bad phenomena. ("flats" Θ-separating an edge but not the simplex.)

How to resolve:

- When constructing vertices of $\mathbf{G}_V$, use good "flats" only. Also mark these flats with good separation types. (Same "flat" with different separation types are viewed as different vertices).
- Use Θ to define edges.
- For any $W \subset V$, $\mathbf{G}_W$ may contain more information than a corresponding subgraph of $\mathbf{G}_V$. ($\mathbf{G}_W$ is an "enrichment" of a subgraph of $\mathbf{G}_V$.) Tedious technical discussions are needed.

After studying the relations between "polygonal pieces" and $\mathbf{G}_V$, one can use the language of $\mathbf{G}_V$ to describe the filling process needed for the construction of higher dimensional straightened simplices.

Thank you!