# Geometric structures, Gromov norm and Kodaira dimensions

Weiyi Zhang

University of Warwick

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# Compact Riemann surfaces

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| Curvature | $>0$ | $0$ | $<0$ |
| Geometry | Spherical | Euclidean | Hyperbolic |
| Canonical bundle | not effective | trivial | ample |
| $\dim H^0(\mathcal{K}^{\otimes l})$ | $0,\ l>0$ | $1,\ l\geq 0$ | $(2l-1)(g-1), l\geq 2$ |
| Kodaira dimension | $-\infty$ | $0$ | $1$ |

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In this talk, all manifolds are closed, smooth and oriented.

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where $s_0, \cdots, s_N \in H^0(\mathcal{K}_J^{\otimes l})$ is a basis of the global sections of $\mathcal{K}_J^{\otimes l}$.

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The Kodaira dimension is the supremum, as $l \to \infty$, of the (complex) dimension of the image of $M$ under these maps. It takes values in $\{-\infty, 0, \cdots, m\}$.

# Equivalent formal definition

**Definition**

Suppose $(M, J)$ is a complex manifold of real dimension $2m$. The holomorphic Kodaira dimension $\kappa^h(M, J)$ is defined as follows:

$$\kappa^h(M,J) = \begin{cases} -\infty & \text{if } \dim H^0(\mathcal{K}_J^{\otimes l}) = 0 \text{ for all } l \geq 1, \\ 0 & \text{if } \dim H^0(\mathcal{K}_J^{\otimes l}) \in \{0,1\}, \text{ but } \not\equiv 0 \text{ for all } l \geq 1, \\ k & \text{if } \dim H^0(\mathcal{K}_J^{\otimes l}) \sim cl^k;\ c > 0. \end{cases}$$

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- Friedman-Qin: If $(X, J_1)$ and $(X, J_2)$ are diffeomorphic complex surfaces, then $\kappa^h(X, J_1) = \kappa^h(X, J_2)$.
- In higher dimensions, there are examples $\kappa^h(X, J_1) \neq \kappa^h(X, J_2)$. For instance, take $(X, J_1) = (M^4 \times \Sigma_{g\geq 2}, J_M \times j)$ and $(X, J_2) = ((M_0 = \mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}) \times \Sigma_{g\geq 2}, J_{M_0} \times j)$, where $M^4$ is the Barlow surface, an algebraic surface of general type homeomorphic but not diffeomorphic to $M_0$.
  Then $\kappa^h(X, J_1) = 3$ and $\kappa^h(X, J_2) = -\infty$.

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$\kappa^h = 1$: Elliptic surfaces, and their blow-ups;

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Geometric reason:
For rational curves $S$ with $S \cdot S \geq -1$, $\mathcal{K}_J|_S$ doesn't admit sections.
For elliptic curves $T$ with $T \cdot T = 0$, $\mathcal{K}_J|_T$ is trivial.

# Symplectic Kodaira dimension for 4-manifolds

## Definition (Li, LeBrun, McDuff-Salamon)

For a minimal (*i.e. no* $-1$ *smooth spheres*) symplectic 4-manifold $(M^4, \omega)$ with symplectic canonical class $K_\omega = c_1(T^*M)$, the Kodaira dimension of $(M^4, \omega)$ is defined in the following way:

$$\kappa^s(M^4,\omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } & K_\omega \cdot K_\omega < 0, \\ 0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } & K_\omega \cdot K_\omega = 0, \\ 1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } & K_\omega \cdot K_\omega = 0, \\ 2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } & K_\omega \cdot K_\omega > 0. \end{cases}$$

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its minimal models.

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- $\kappa^s$ is a smooth invariant. (Taubes)
- When $M^4$ admits both complex and symplectic structures, $\kappa^h = \kappa^s$. (Dorfmeister-Z.)
- If $\kappa^s(M) = -\infty$, then $M$ is rational or ruled. (A.-K. Liu)
- For minimal $M$, $\kappa^s(M) = 0 \Leftrightarrow K_\omega$ is torsion. And $M$ has the same $\mathbb{Q}$-homology as a K3, Enriques, or $T^2$ bundle over $T^2$. (Li)

# Additivity

We can also define Kodaira dimension for 0D and 1D.
0D: points, $Kod = 0$; 1D: circles, $Kod = 0$.

Philosophy: Given some "fibration/pencil" $f : M \to N$, then

$$Kod(M) = Kod(N) + Kod(fiber).$$

The Kodaira dimensions might be a relative one.

## Examples

- Covering. If $f : M \to N$ is a finite unramified covering, Then $\kappa^h(M) = \kappa^h(N)$ (Ueno), $\kappa^s(M) = \kappa^s(N)$ (Li-Z.).

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- Maps with non-zero degree. If $f : M \to N$ is a map of non-zero degree, we should expect $Kod(M) \geq Kod(N)$, *i.e.*, compatible with Gromov's order.
- Do we have $Kod(M^3 \times S^1) = Kod_3(M^3) + Kod(S^1) = Kod_3(M^3)$?
  Need a proper definition of $Kod_3(M^3)$ first.

# Geometric structures

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A *geometric structure* on a manifold $M$ is a diffeomorphism from $M$ to $X/\Gamma$ for some model geometry $(X, G)$, where $\Gamma$ is a discrete subgroup of $G$ acting freely on $X$.

# Model geometry

- $X$ simply connected smooth manifold;
- Lie group $G$ acts on $X$ transitively and effectively with compact point stabilizers;
- $G$ is maximal among such groups;
- There is at least one $M$ diffeomorphic to $X/\Gamma$ with finite volume, where $\Gamma$ is a discrete subgroup of $G$ acting freely on $X$.

First two conditions guarantee left $G$-invariant Riemannian metric on $X$. The third implies $G = Isom(X)$.

# Geometric structures in dimension 3

The fourth condition above implies there are finitely many geometric structures. In fact, in dimension 3, there are 8 geometries. We divide them into 4 classes, called "*cat*".

$$\begin{aligned} -\infty: &\quad S^3 \text{ and } S^2\times\mathbb{E};\\ 0: &\quad \mathbb{E}^3, \mathit{Nil} \text{ and } \mathit{Sol};\\ 1: &\quad \mathbb{H}^2\times\mathbb{E}, \widetilde{SL_2(\mathbb{R})};\\ \tfrac{3}{2}: &\quad \mathbb{H}^3. \end{aligned}$$

# Examples

Example of *Nil*: quotient of Heisenberg group by the "integral Heisenberg group".

Example of *Sol*: mapping torus of an Anosov map of the 2-torus, *e.g.* the automorphism given by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Examples of $\widetilde{SL_2(\mathbb{R})}$: unit tangent bundle of $\Sigma_{g\geq 2}$; Brieskorn homology spheres $\{x^p + y^q + z^r = 0\} \cap S^5$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.

# Geometrization

To define the Kodaira dimension for a general 3-manifold, we need to use Thurston's geometrization.

**Theorem (Kneser-Milnor)**

*Every compact, orientable 3-manifold can be decomposed into the connected sum of a unique (finite) collection of prime 3-manifolds.*

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*Every compact, orientable 3-manifold can be decomposed into the connected sum of a unique (finite) collection of prime 3-manifolds.*

**Theorem (Perelman, Hamilton)**

*Every oriented prime closed 3-manifold can be cut along tori, so that the interior of each of the resulting manifolds has a geometric structure with finite volume.*

So start with any closed 3-manifold, we can decompose along spheres and tori such that each resulting piece is geometric.

# Definition of $\kappa^t(M^3)$

**Definition (and Proposition)**

$\kappa^t(M^3) = \max_i\{cat(M_i) \mid M_i$ are geometric pieces$\}$ is well defined.

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**Theorem (Thurston)**

*Non-closed 3-dimensional geometric manifolds with finite volume exist only for geometries in category 1 or $\frac{3}{2}$, i.e. $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}$ and $\widetilde{SL_2(\mathbb{R})}$.*

# Definition of $Kod_3$ and virtual Betti number

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We define the virtual Betti number

$$vb_1(M) := \sup\{b_1(\tilde{M}) \mid \tilde{M} \text{ is a finite covering of } M\}.$$

Then for irreducible 3-manifolds, there is a numerical characterization of $\kappa^t(M^3)$:

$$Kod_3(M^3) = \begin{cases} -\infty & \text{when } vb_1 = 0, \\ 0 & \text{when } vb_1 \text{ is finite and positive,} \\ 1 & \text{when } vb_1 \text{ is infinite.} \end{cases}$$

# Compatiblity

- When $M^3 \times S^1$ is complex/symplectic, $M$ is a $\Sigma_g$ bundle over $S^1$ (Friedl-Vidussi, Etgü). Then it is easy to check $Kod_3(M^3) = [\kappa^t(M^3)] = Kod(M^3 \times S^1)$.
- It is also compatible with 2-d, 1-d Kodaira dimension in the sense of additivity.
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- It is also compatible with 2-d, 1-d Kodaira dimension in the sense of additivity.
- It is invariant under unramified covering.

Furthermore, we have

**Theorem**

*If $f : M^3 \longrightarrow N^3$ is a non-zero degree map, then $\kappa^t(M) \geq \kappa^t(N)$.*

One key point: geometric manifolds are determined by their fundamental groups.

# Definition of Gromov norm

To show $\mathbb{H}^3$ is the largest, we use Gromov norm.
Let $|\cdot|_1 : C_k(M;\mathbb{R}) \to \mathbb{R}$ be the $l^1$ norm on real singular chains: for $z = \sum c_i\sigma_i \in C_k(M;\mathbb{R})$,

$$|z|_1 := \sum |c_i|.$$

Then the *Gromov norm* is

$$||M|| := \inf\{|z|_1 \mid [z] = [M]\} \in \mathbb{R}_{\geq 0}.$$

# Basic properties

- By definition, $||M|| \geq \deg(f)||N||$, if $f: M^n \to N^n$.
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- For hyperbolic manifold $M^n$, $||M|| = \frac{Vol(M)}{v_n} > 0$, where $v_n = Vol(\text{regular ideal simplex})$, *e.g.* $v_2 = \pi$.
- Gluing. The Gromov norm is additive when gluing along spheres or tori.

# More vanishing results

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$\Rightarrow$ A 3-manifold has vanishing Gromov norm if and only if it is a graph manifold.
All the above imply

**Lemma**

*If $f : M^3 \to N^3$ with $\deg f \neq 0$, then $\kappa^t(N) = 1.5$ implies $\kappa^t(M) = 1.5$.*

# 19 geometries in dimension 4

The classification of geometric structures in dimension 4 due to Filipkiewicz (Warwick thesis)

$-\infty$ : $\mathbb{P}^2(\mathbb{C})$, $S^4$, $S^3\times\mathbb{E}$, $S^2\times S^2$, $S^2\times\mathbb{E}^2$, $S^2\times\mathbb{H}^2$, $Sol_0^4$ and $Sol_1^4$;
$0$ : $\mathbb{E}^4$, $Nil^4$, $Nil^3\times\mathbb{E}$ and $Sol^4_{m,n}$(including $Sol^3\times\mathbb{E}$);
$1$ : $\mathbb{H}^2\times\mathbb{E}^2$, $\widetilde{SL_2}\times\mathbb{E}$, and $F^4$;
$\frac{3}{2}$ : $\mathbb{H}^3\times\mathbb{E}$;
$2$ : $\mathbb{H}^2(\mathbb{C})$, $\mathbb{H}^2\times\mathbb{H}^2$ and $\mathbb{H}^4$.

Here $F^4$ admits no compact model.

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0:&\quad \mathbb{E}^4,\ Nil^4,\ Nil^3\times\mathbb{E} \text{ and } Sol_{m,n}^4 (\text{including } Sol^3\times\mathbb{E});\\
1:&\quad \mathbb{H}^2\times\mathbb{E}^2,\ \widetilde{SL_2}\times\mathbb{E}, \text{ and } F^4;\\
\tfrac{3}{2}:&\quad \mathbb{H}^3\times\mathbb{E};\\
2:&\quad \mathbb{H}^2(\mathbb{C}),\ \mathbb{H}^2\times\mathbb{H}^2 \text{ and } \mathbb{H}^4.
\end{aligned}$$

Here $F^4$ admits no compact model.

**Definition**

Let $M^4$ be a 4-dimensional geometric manifold. The Kodaira dimension $\kappa^g(M)$ is defined to be the category number of $M$.

# Geometric manifolds of $[\kappa^g] = 1$

**Theorem**

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A geometric manifold with $\kappa^g = -\infty$ or 0 has amenable fundamental group or admits a non trivial $S^1$ action. In particular, it has vanishing Gromov norm.

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**Corollary**

*A closed geometric 4-manifold has nonzero Gromov norm if and only if* $\kappa^g = 2$.

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The "if" part of the above corollary follows from

- $\mathbb{H}^4$: $||M|| = \frac{1}{v_4} Vol(M)$ (Gromov-Thurston)
- $\mathbb{H}^2 \times \mathbb{H}^2$: $||M|| = \frac{3}{2\pi^2} Vol(M)$. (Bucher-Karlsson)
- $\mathbb{H}^2(\mathbb{C}) = SU(2,1)/S(U(2) \times U(1))$ is a closed oriented locally symmetric space of non-compact type, then $||M|| > 0$ (Lafont-Schmidt).

# Mapping order of Geometric 4-manifolds

It is easy to see that $\kappa^g$ is preserved under finite covering.
Similar to the result for monotonicity of $\kappa^t$ for 3-manifolds, we have

**Theorem (Neofytidis)**

*If $f : M^4 \longrightarrow N^4$ is a map of non-zero degree between closed geometric 4-manifolds, then $\kappa^g(M) \geq \kappa^g(N)$.*

# Hitchin-Thorpe

As suggested by the solution of geometrization conjecture by Ricci flow, building blocks of 4-manifolds should consist of Einstein manifolds and collapsed pieces. Einstein 4-manifolds are far more complicated than Einstein 3-manifolds. However, we have the following Hitchin-Thorpe theorem.

**Theorem (Hitchin-Thorpe)**

*Any compact oriented Einstein 4-manifold $(M, g)$ satisfies $2\chi + 3\sigma \geq 0$. The equality holds if and only if $(M, g)$ is finitely covered by a Calabi-Yau K3 surface or by a 4-torus.*

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For symplectic/complex 4-manifolds with $Kod = 1$, we have $K^2 \leq 0$ since blow-up reduces $K^2$. By Hitchin-Thorpe, such Einstein ones have $K^2 = 0$, which holds for $Kod = 0$.

# No symplectic/complex Einstein manifolds with $Kod = 1$

If there is an almost complex structure $J$, then $2\chi + 3\sigma = K^2$.

For symplectic/complex 4-manifolds with $Kod = 1$, we have $K^2 \leq 0$ since blow-up reduces $K^2$. By Hitchin-Thorpe, such Einstein ones have $K^2 = 0$, which holds for $Kod = 0$.

Hence there is no symplectic/complex Einstein manifolds with $Kod = 1$.

This suggests contributions from Kodaira dimension 1 are all from collapsed pieces. All from Model Geometries?

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- It coincides with Gromov order. That is, if $f : M^4 \longrightarrow N^4$ is a map of non-zero degree between closed geometric 4-manifolds, then $\kappa^g(M) \geq \kappa^g(N)$.
- A closed geometric 5-manifold has nonzero Gromov norm if and only if $\kappa^g = \frac{5}{2}$, *i.e.*, when $M$ is modeled on $\mathbb{H}^5, SL(3,\mathbb{R})/SO(3)$ or $\mathbb{H}^3 \times \mathbb{H}^2$.

# Higher dimensional geometric manifolds

A complete connected Riemannian manifold $M$ is a *locally symmetric space of non-compact type* if it is diffeomorphic to $\Gamma\backslash G/K$ where $G$ is a centerless semisimple Lie group, $K$ is a maximal compact subgroup in $G$, and $\Gamma$ is a discrete subgroup in $G$ that acts freely on $G/K$ via left action.

**Theorem (P. H. How)**

*Let $M$ be a geometric manifold. Then $||M|| > 0$ if and only if $M$ is a locally symmetric space of non-compact type.*

We can merely assume $M$ is a closed topological manifold that admits a smooth structure with a locally homogeneous Riemannian metric.

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- Let $M$ be a smooth 4-dimensional symplectic manifold with nonvanishing Gromov norm. Is $\kappa^s(M) = 2$?
  This is true for Kähler surfaces.
- Suppose that $(M_1, \omega_1)$ and $(M_2, \omega_2)$ are symplectic 4-manifolds and almost complex structures $J_i$ are tamed by $\omega_i$. If $f$ is a $(J_1, J_2)$-pseudo-holomorphic map (*i.e.* $f \circ J_1 = J_2 \circ f$) of non-zero degree from $(M_1, \omega_1)$ to $(M_2, \omega_2)$, is $\kappa^s(M_1, \omega_1) \geq \kappa^s(M_2, \omega_2)$?
  Note: If there is a holomorphic map of non-zero degree from $(M_1, J_1)$ to $(M_2, J_2)$, then $\kappa^h(M_1, J_1) \geq \kappa^h(M_2, J_2)$.

# Answer to Question 1: Kähler 3-folds

We have a satisfactory answer to above Question 1: Let $M$ be a smooth $2n$-dimensional complex manifold with nonvanishing Gromov norm. Is $\kappa^h(M) = n$?

**Theorem (Neofytidis-Z.)**

*If $X$ is a smooth Kähler 3-fold with non-vanishing Gromov norm, then $\kappa^h(X) = 3$.*

We can reduce it to projective manifolds, since for any compact Kähler manifold $X$ of complex dimension three, there exists a bimeromorphic Kähler manifold $X'$ which is deformation equivalent to a projective manifold.

# Answer to Question 1: Projective manifolds

It is based on the following facts

- Gromov norm is a birational invariant. That is, birationally equivalent smooth projective varieties (resp. bimeromorphic smooth Kähler manifolds) have the same Gromov norm.
- Any uniruled manifold has vanishing Gromov norm.

and two conjectures, proved in dimension $\leq 3$:

**Conjecture**

1. *(Mumford) A smooth projective variety with* $\kappa^h = -\infty$ *is uniruled.*
2. *(Kollár) Let* $X$ *be a smooth projective variety with* $\kappa^h(X) = 0$*. Then* $\pi_1(X)$ *has a finite index Abelian subgroup.*

# Answer to Question 1: Projective manifolds (cont.)

## Theorem (Neofytidis-Z.)

- *Up to Kollár's Conjecture, any smooth 2n-dimensional complex projective variety M with* $\kappa^h(M) \geq 0$ *and* $\|M\| > 0$ *has* $\kappa^h(M) = n$.
- *Further assuming Mumford's Conjecture, any smooth projective variety with non-vanishing Gromov norm is of general type.*

## Corollary

*Let M be a smooth complex projective n-fold with non-vanishing simplicial volume. Then* $\kappa^h(M)$ *cannot be* $n-1, n-2$ *or* $n-3$. *If, moreover,* $n = 3$, *then* $\kappa^h(M) = 3$.

# Yamabe invariant

$$Y(M) = \sup_{[\hat{g}]\in\mathcal{C}} \inf_{g\in[\hat{g}]} \int_M s_g dV_g,$$

where $g$ is a Riemannian metric on $M$, $s_g$ is the scalar curvature of $g$, and $\mathcal{C}$ is the set of conformal classes on $M$.

**Question (LeBrun)**

*If $M^4$ admits a symplectic structure and $Y(M^4) < 0$, is $\kappa^s(M^4) = 2$?*

- For minimal $M^4$, $\kappa^s(M^4) = 2$ would imply $Y(M^4) < 0$.
- It is true for Kähler surfaces.
- F-structure $\Rightarrow Y(M^4) \geq 0$. $Y(M) > 0$ if and only if $M$ has positive scalar curvature.

# Pluricanonical genus

We can generalize the two (equivalent) definitions of complex Kodaira dimension to almost complex manifolds.

We still have $\mathcal{K} = \Lambda^{n,0}$, but it is no longer holomorphic. However, we have $\bar\partial : \mathcal{K} \to \Lambda^{n,1} \cong T^{0,1} \otimes \mathcal{K}$. and it can be extended to an operator $\bar\partial_m : \mathcal{K}^{\otimes m} \to T^{0,1} \otimes \mathcal{K}^{\otimes m}$ for $m \geq 2$ inductively by Leibniz rule.

We still have the pluricanonical genus

$$P_m(X, J) = H^0(X, \mathcal{K}^{\otimes m}) = \{s \in \Gamma(X, \mathcal{K}^{\otimes m}) : \bar\partial_m s = 0\}.$$

**Proposition**

*$H^0(X, \mathcal{K}^{\otimes m})$ is finite dimensional.*

# Kodaira dimension $\kappa^J(X)$

**Definition**

The Kodaira dimension $\kappa^J(X)$ of $(X, J)$ is defined as:

$$\kappa^J(X) = \begin{cases} -\infty, \text{ if } P_m = 0 \text{ for any } m \geq 0 \\ \limsup_{m\to\infty} \dfrac{\log P_m}{\log m}, \text{ otherwise.} \end{cases}$$

# Kodaira dimension $\kappa_J(X)$

We can still define the pluricanonical map by

$$\Phi_m(x) = [s_0(x) : \cdots : s_N(x)],$$

where $s_i$ are a basis of $H^0(X, \mathcal{K}^{\otimes m})$.

**Theorem**

*$\Phi_m$ is a pseudoholomorphic map and the image is a projective subvariety of $\mathbb{C}P^N$.*

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We only summarize the results for almost complex 4-manifolds.

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- $\kappa^J$ is not a deformation invariant, hence not a smooth invariant.
- $\kappa^J = -\infty$ or 0, if $J$ is not integrable.
- We have similar generalizations of Iitaka dimension.

Thanks very much for your attention !